HOMOGENEOUS COORDINATES

FOR

USE IN COLLEGES AND SCHOOLS

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PREFACE

IN the present treatise, I have attempted to present the subject of Homogeneous Coordinates (particular attention being paid to Areals and Trilinears) in as concise a form as is consistent with clearness. In writing the work, I have kept continually in view the needs of Honours Students at the Universities and of pupils specialising in Mathematics at Secondary Schools. The book is intended to teach processes and methods, rather than results, and I have, therefore, in those cases explicitly stated in black type *without proof* geometrical results which belong more properly to the province of Pure Geometry, and in an explanatory note (p. xi) have given references to other works in which demonstrations may be found. In this way a considerable amount of ground has been covered and the book has not been made unduly long.

It is hoped that the present treatise will provide a suitable introduction to such works as Salmon's *Conics and Higher Plane Curves*, which are usually found rather hard reading by students who approach them for the first time without some previous grounding in the methods of Homogeneous Coordinates.

The method of treatment throughout has been made as geometrical as is possible in a book intended primarily to teach analytical methods. The principles and processes of Tangential Coordinates receive full explanation, and the analogy between the results obtained for line-coordinates and the results previously obtained for point-coordinates is at every stage carefully pointed out.

The tangential equation to the Circular Points at Infinity is found at the outset in Chapter IV. in dealing with Circles, Rectangular Hyperbolas, Foci etc., and thus, by treating the Circular Points as a degenerate conic, many previous results in conics can be at once quoted. In this way much memory work is avoided. The last chapter on Parametric Representation will, it is hoped, be found particularly useful, especially by those who intend to study the more advanced portions of geometry.

The examples have for the most part been collected from the Scholarship Papers set at the various Colleges in Cambridge, and I have to thank the Syndics of the University Press and the various College Tutors for their kind permission to make use of them.

I wish also to express my indebtedness to J. H. Grace, Esq., F.R.S., formerly Fellow of St Peter's College, Cambridge; H. C. Beaven, Esq., M.A., Head of the Mathematical Department, Clifton College; Peter Fraser, Esq., M.A., B.Sc., Lecturer in Mathematics at the University of Bristol; F. C. Stephen, Esq., M.A., Emmanuel College, Cambridge, for kindly criticising the work while in manuscript form and giving many valuable suggestions. I am also under great obligations to Dr Charles McLeod, Senior Mathematical Master, Aberdeen Grammar School, who not only read the work while in manuscript but also revised the proofs while it was passing through the press.

I should be deeply grateful for any corrections or suggestions from those who may use the book.

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Clifton College.

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EXPLANATORY NOTE

In the following list of references to the geometrical theorems stated without proof in black type throughout the text, the abbreviation Proj. Geom. refers to the work entitled "Projective Geometry" by L. N. G. Filon, M.A., D.Sc., (Arnold), and Pure Geom. to the work entitled "An Elementary Treatise on Pure Geometry" by John Wellesley Russell, M.A., (Clarendon Press).

1. Modern Analytical Geometry by Charlotte Angas Scott, D.Sc., Chap. II. Art. 26.
2. Salmon's Higher Plane Curves, Chap. V. Art. 165.
3. Proj. Geom. Chap. II. Art. 20; Pure Geom. Chap. IX. Art. 7.
4. Proj. Geom. Chap. II. Art. 28; Pure Geom. Chap. II. Art. 1.
5. Proj. Geom. Chap. II. Art. 29; Pure Geom. Chap. II. Art. 12.
6. Proj. Geom. Chap. II. Art. 29; Pure Geom. Chap. II. Art. 10.
7. Proj. Geom. Chap. IV. Arts. 47, 48, 60; Pure Geom. Chap. III. Art. 7 and Chap. V. Art. 5.
8. Proj. Geom. Chap. IV. Art. 51; Pure Geom. Chap. III. Art. 10.
9. Proj. Geom. Chap. IV. Art. 59; Pure Geom. Chap. III. Art. 14.
10. Proj. Geom. Chap. XII. Art. 200; Pure Geom. Chap. XI. Art. 7.
11. Proj. Geom. Chap. IV. Art. 57; Pure Geom. Chap. V. Art. 11.
12. Compare Conic Sections by C. Smith, M.A., Chap. XI. Art. 230; Pure Geom. Chap. XI. Art. 7.
13. Proj. Geom. Chap. XII. Art. 207; Differential Calculus by Joseph Edwards, M.A., Chap. X. Arts. 350 *et seq.*
14. Compare Geometrical Conics by C. Smith, M.A., Chap. VI. Art. 144; Proj. Geom. Chap. V. Art. 86 and Chap. VI. Art. 109.
15. Differential Calculus by Joseph Edwards, M.A., Chap. X. Arts. 354 and 356.
16. Proj. Geom. Chap. III. Art. 34; Pure Geom. Chap. V. Art. 2.
17. Proj. Geom. Chap. III. Art. 34; Pure Geom. Chap. V. Art. 2.

18. Proj. Geom. Chap. III. Art. 34; Pure Geom. Chap. V. Art. 2.
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20. Proj. Geom. Chap. IV. Art. 53; Pure Geom. Chap. III. Art. 12.
21. Proj. Geom. Chap. IV. Art. 51; Pure Geom. Chap. III. Art. 12.
22. Proj. Geom. Chap. XII. Art. 197; Pure Geom. Chap. XII. Art. 6.
23. Proj. Geom. Chap. XII. Art. 200; Pure Geom. Chap. XII. Art. 6.
24. Proj. Geom. Chap. XII. Art. 197; Pure Geom. Chap. XII. Art. 6.
25. Proj. Geom. Chap. X. Art. 164; Pure Geom. Chap. XXVIII. Arts. 3 and 5.
26. Proj. Geom. Chap. X. Art. 165; Pure Geom. Chap. XXVIII. Art. 1.
27. Proj. Geom. Chap. X. Art. 167; Pure Geom. Chap. XXVIII. Art. 2.
28. Proj. Geom. Chap. X. Art. 169; Pure Geom. Chap. XXVIII. Art. 8.
29. Proj. Geom. Chap. III. Art. 37; Pure Geom. Chap. XI. Art. 2.

HOMOGENEOUS COORDINATES

CHAPTER I

THE STRAIGHT LINE

1. *Areal Coordinates.*

Let three given non-concurrent straight lines form the triangle ABC (Fig. 1). Let P be a point whose

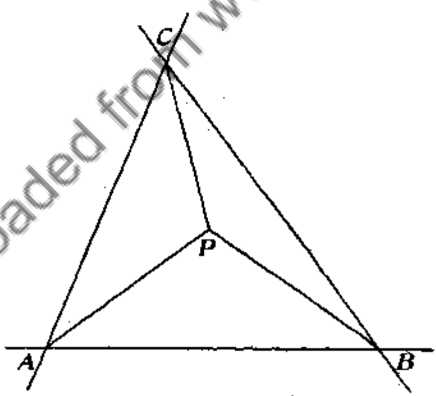

Fig. 1.

position we wish to define with respect to the triangle ABC. We shall regard P as lying on the *positive* side of

BC if P lies on the same side of BC as A. Similarly we shall regard P as lying on the *positive* side of CA if it lies on the same side of CA as B, and similarly with regard to the side AB. The ratios of the areas of the triangles PBC, PCA, PAB to the area of the triangle ABC are called the "*areal coordinates*" of the point P, and are denoted thus ;—

$$x = \frac{\text{Area of } \triangle\ PBC}{\text{Area of } \triangle\ ABC};$$

$$y = \frac{\text{Area of } \triangle\ PCA}{\text{Area of } \triangle\ ABC};$$

$$z = \frac{\text{Area of } \triangle\ PAB}{\text{Area of } \triangle\ ABC}.$$

For shortness of notation we write $P \equiv (x, y, z)$ to denote that the coordinates of P are x, y, z respectively. We see at once that

$$A \equiv (1, 0, 0);\ B \equiv (0, 1, 0);\ C \equiv (0, 0, 1).$$

We shall always denote the vertices of the triangle of references by the letters A, B, C and the corresponding opposite sides by a, b, c respectively.

2. *To find the identical relation existing between the three areal coordinates of a given point.*

It is to be noted that of the *three* areal coordinates of a point, only *two* are required to define its position uniquely. For suppose that x and y are given, then the areas of the triangles PBC and PCA are given and hence P will lie on each of two determinate lines parallel to BC and CA respectively, i.e. at their point of intersection. Thus if x and y are given, z can be found at once and hence a relationship must exist between x, y and z.

CASE I.

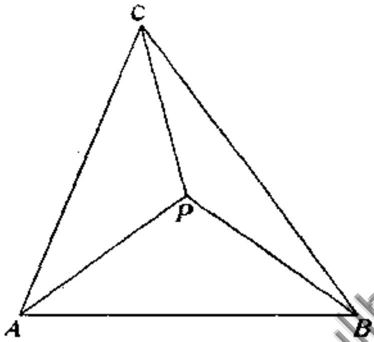

Fig. 2.

Let P lie inside the triangle of reference (Fig. 2). Then

$$x = +\frac{\text{Area of } \triangle\ PBC}{\text{Area of } \triangle\ ABC};\quad y = +\frac{\text{Area of } \triangle\ PCA}{\text{Area of } \triangle\ ABC};$$

$$z = +\frac{\text{Area of } \triangle\ PAB}{\text{Area of } \triangle\ ABC}.$$

$$\therefore \quad x + y + z = 1.$$

CASE II.

Next let P lie outside the triangle of reference between CA and BC but on the side of AB remote from C (Fig. 3). Then remembering our convention of Art. 1 as to the negative side of AB we have

$$x = +\frac{\text{Area of } \triangle\ PBC}{\text{Area of } \triangle\ ABC};\quad y = +\frac{\text{Area of } \triangle\ PCA}{\text{Area of } \triangle\ ABC};$$

$$z = -\frac{\text{Area of } \triangle\ PAB}{\text{Area of } \triangle\ ABC}.$$

$$\therefore \quad x + y + z = 1.$$

A corresponding proof holds if P occupy similar positions with regard to A and B.

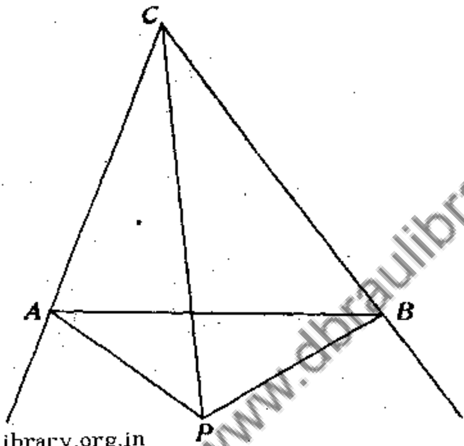

Fig. 3.

CASE III.

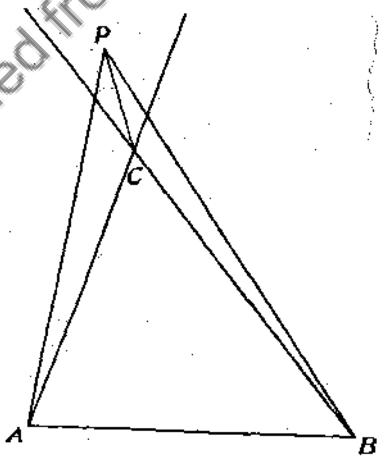

Fig. 4.

Let P lie outside the triangle of reference between BC and CA but on the same side of AB as C (Fig. 4). Then

$$x = -\frac{\text{Area of } \triangle\ PBC}{\text{Area of } \triangle\ ABC};\quad y = -\frac{\text{Area of } \triangle\ PCA}{\text{Area of } \triangle\ ABC};$$

$$z = +\frac{\text{Area of } \triangle\ PAB}{\text{Area of } \triangle\ ABC}.$$

$$\therefore\quad x + y + z = 1.$$

A corresponding proof holds if P occupy similar positions with regard to A and B.

Thus for all positions of P not at infinity, the areal coordinates of P satisfy the identical relationship

$$x + y + z = 1.$$

3. *Actual Areal Coordinates.*

In finding the position of a point it is not necessary to be given the actual values of the areal coordinates, if only we be given numbers *proportional* to them. Thus suppose that the numbers 2, 3, 4 are proportional to the actual areal coordinates of a point. Let the actual areal coordinates be 2λ, 3λ, 4λ. Then by the fundamental identity of Art. 2

$$2\lambda + 3\lambda + 4\lambda = 1.$$

$$\therefore\quad \lambda = \tfrac{1}{9}.$$

Hence the actual areal coordinates are $\frac{2}{9}, \frac{3}{9}, \frac{4}{9}$.

We shall in general use numbers x', y', z' only proportional to the actual areal coordinates. Thus the "actual areal coordinates" of the point $P \equiv (x', y', z')$ are

$$\frac{x'}{x' + y' + z'},\quad \frac{y'}{x' + y' + z'},\quad \frac{z'}{x' + y' + z'}.$$

4. *Homogeneity in Equations.*

We can always render an algebraic equation homogeneous in x, y, z by means of the identity $x+y+z=1$.

Thus $2x^2-4y+3=0$ becomes

$$2x^2-4y(x+y+z)+3(x+y+z)^2=0.$$

5. *Trilinear Coordinates.*

The "*trilinear coordinates*" of the point P are defined to be the three perpendiculars dropped from P on the three sides of the triangle of reference, the same convention as to signs holding as in areal coordinates. If α, β, γ be the actual trilinear coordinates of a point and x, y, z the actual areal coordinates of the same point and if Δ denote the area of the triangle of reference, then we can transform from one system to the other by means of the formulae

$$x=\frac{a_0\alpha}{2\Delta},$$

$$y=\frac{b_0\beta}{2\Delta},$$

$$z=\frac{c_0\gamma}{2\Delta},$$

where a_0, b_0, c_0 denote respectively the lengths of the sides of the triangle of reference. Then plainly the identical relationship connecting the trilinear coordinates of any point not at infinity is

$$a_0\alpha+b_0\beta+c_0\gamma=2\Delta.$$

6. *If x_1, y_1, z_1 be the actual areal coordinates of P_1 and x_2, y_2, z_2 the actual areal coordinates of P_2, to find the actual areal coordinates of a point P dividing the line P_1P_2 in the ratio $m_2:m_1$.*

Drop the perpendiculars P_1N_1 and P_2N_2 to AB. Through P draw U_1PU_2 parallel to AB (Fig. 5).

Then
$$\frac{m_2}{m_1}=\frac{P_1U_1}{U_2P_2}=\frac{PN-P_1N_1}{P_2N_2-PN}$$
$$=\frac{\frac{c_0\,.\,PN}{2\Delta}-\frac{c_0\,.\,P_1N_1}{2\Delta}}{\frac{c_0\,.\,P_2N_2}{2\Delta}-\frac{c_0\,.\,PN}{2\Delta}}$$
$$=\frac{z-z_1}{z_2-z}.$$

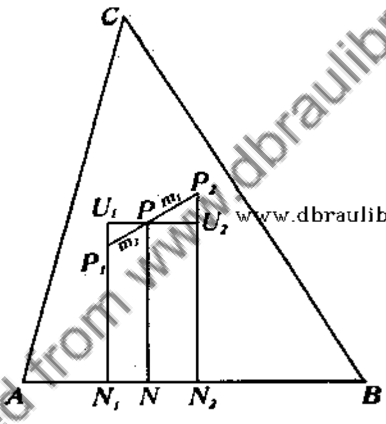

Fig. 5.

Hence
$$z=\frac{m_1z_1+m_2z_2}{m_1+m_2}.$$

So for x and y.

Thus in actual areal coordinates
$$P\equiv\left(\frac{m_1x_1+m_2x_2}{m_1+m_2},\ \frac{m_1y_1+m_2y_2}{m_1+m_2},\ \frac{m_1z_1+m_2z_2}{m_1+m_2}\right),$$
but as we are in general concerned only with numbers proportional to the actual areal coordinates, we may write more briefly $P\equiv(m_1x_1+m_2x_2,\ m_1y_1+m_2y_2,\ m_1z_1+m_2z_2)$.

1°. N.B. This formula has been proved on the assumption that x_1, y_1, z_1 and x_2, y_2, z_2 are *each* the actual areal coordinates of P_1 and P_2 respectively, but plainly P will still be the same point if we use the *same* multiples of the actual areal coordinates, e.g. $(\lambda x_1, \lambda y_1, \lambda z_1)$ and $(\lambda x_2, \lambda y_2, \lambda z_2)$. For P would then become

$$P \equiv (m_1\lambda x_1 + m_2\lambda x_2,\ m_1\lambda y_1 + m_2\lambda y_2,\ m_1\lambda z_1 + m_2\lambda z_2),$$

which is plainly the same point as before. But, if we used different multiples, e.g. λ and μ, thus getting $P_1 \equiv (\lambda x_1, \lambda y_1, \lambda z_1)$ and $P_2 \equiv (\mu x_2, \mu y_2, \mu z_2)$, and inserted these in the formula for P, the above formula would not then represent the same point P.

2°. This is one of the very few formulae where such a restriction has to be made. In most formulae

$$(\lambda x_1, \lambda y_1, \lambda z_1), (\mu x_2, \mu y_2, \mu z_2), (\nu x_3, \nu y_3, \nu z_3) \text{ etc.},$$

where (x_1, y_1, z_1) etc. are the actual areal coordinates of the points in question, may be used with impunity.

7. *To find the equation of the straight line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2).*

By last article, if x, y, z be the actual areal coordinates of the point dividing the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_2 : m_1$, then

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2},\quad y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2},\quad z = \frac{m_1z_1 + m_2z_2}{m_1 + m_2}.$$

Eliminating $\dfrac{m_1}{m_1 + m_2}$ and $\dfrac{m_2}{m_1 + m_2}$ we get as the equation required to the given line

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

We note also that we may write the above equation as

$$\begin{vmatrix} \lambda x & \lambda y & \lambda z \\ \lambda_1 x_1 & \lambda_1 y_1 & \lambda_1 z_1 \\ \lambda_2 x_2 & \lambda_2 y_2 & \lambda_2 z_2 \end{vmatrix} = 0,$$

where λ, λ_1, λ_2 are any numerical multipliers.

Hence in this formula, "actual areal coordinates" need not be used.

COROLLARY.

The equation to any given straight line is of the first degree in x, y, z.

8. *To find the general equation to a straight line passing through the point of intersection of two given straight lines.*

Let the two straight lines be

$$l_1x + m_1y + n_1z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$l_2x + m_2y + n_2z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Consider the line

$$(l_1x + m_1y + n_1z) + \lambda(l_2x + m_2y + n_2z) = 0 \quad\ldots(3),$$

where λ is a numerical multiplier. Since (3) is of the first degree in x, y, z it represents a straight line. Also the coordinates of the point of intersection of the given two lines satisfy (1) and (2) and, therefore, satisfy (3). Hence the line (3) passes through the point of intersection of (1) and (2).

The constant λ can be determined so as to make the line (3) satisfy *one* other condition.

9. *To find the condition that three points be collinear.*

Let the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) all lie on the line

$$lx + my + nz = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then $$lx_1 + my_1 + nz_1 = 0 \quad \text{..............(2)},$$

$$lx_2 + my_2 + nz_2 = 0 \quad \text{..............(3)},$$

$$lx_3 + my_3 + nz_3 = 0 \quad \text{..............(4)}.$$

Hence eliminating l, m, n from (2), (3), (4) we get as the condition required

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

It is perfectly easy to prove in a similar way that *the condition that the three lines*

$$l_1x + m_1y + n_1z = 0,$$

$$l_2x + m_2y + n_2z = 0,$$

$$l_3x + m_3y + n_3z = 0$$

are concurrent is

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

10. *To find the coordinates of the point in which the two lines*

$$l_1x + m_1y + n_1z = 0,$$

$$l_2x + m_2y + n_2z = 0$$

meet.

Solve for $x : y : z$ getting

$$\frac{x}{m_1n_2 - m_2n_1} = \frac{y}{n_1l_2 - n_2l_1} = \frac{z}{l_1m_2 - l_2m_1}.$$

Hence the point of intersection

$$\equiv (m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1).$$

11. The Line at Infinity. It can be proved that in any system of Homogeneous Coordinates (of which Areal and Trilinear Coordinates are only particular cases), all points at infinity are to be regarded as lying on a straight line (called "the line at infinity")[1].

To find the equation of the Line at Infinity in areal coordinates.

The point at infinity on BC will be that point L which divides BC in the ratio of $1 : -1$.

Now $B \equiv (0, 1, 0)$ and $C \equiv (0, 0, 1)$ in actual areal coordinates.

Hence $L \equiv (0, -1, 1)$ in areal coordinates (Chap. I, Art. 6) as is easily seen, if we use

$$P \equiv (m_1x_1 + m_2x_2,\ m_1y_1 + m_2y_2,\ m_1z_1 + m_2z_2).$$

Similarly

the point at infinity on CA viz. $M \equiv (1, 0, -1)$

and the point at infinity on AB viz. $N \equiv (-1, 1, 0)$.

Let the equation to the line at infinity be

$$lx + my + nz = 0 \quad \text{.................(1).}$$

Since L lies on this line

$$\therefore \quad -m + n = 0.$$

Hence plainly $\quad l = m = n \quad$ ....(2)

and the equation to the "Line at Infinity" will be

$$x + y + z = 0.$$

COROLLARY.

The Trilinear equation to the Line at Infinity will be

$$a_0\alpha + b_0\beta + c_0\gamma = 0.$$

12. *To find the condition that the two lines*

$$l_1x + m_1y + n_1z = 0 \quad \text{.................(1),}$$

$$l_2x + m_2y + n_2z = 0 \quad \text{.................(2)}$$

be parallel.

Since two parallel lines meet at infinity, we must state the condition that the lines (1) and (2) and the line at infinity are concurrent, i.e.

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

13. We shall solve some Examples.

EXAMPLE 1.

The lines joining the vertices of a triangle ABC to the mid-points of the opposite sides are concurrent.

The mid-point of BC is $(0, 1, 1)$. Hence the line joining A to the mid-point of BC is $y = z$. It is obvious that the lines $y = z$, $z = x$, $x = y$ all meet in the point $(1, 1, 1)$.

EXAMPLE 2.

The straight line joining the mid-points of two sides of a triangle is parallel to the base.

The mid-point of CA is $(1, 0, 1)$.

The mid-point of BC is $(0, 1, 1)$.

The line joining these two points is $x + y - z = 0$ which may be written $x + y + z = 2z$. By Chap. I, Art. 8 this represents a line through the intersection of the line at infinity and AB, i.e. a line parallel to AB.

EXAMPLE 3.

If $O \equiv (x', y', z')$ be a point and if D, E, F be the points in which AO, BO, CO meet the opposite sides respectively, then the points which are the intersections of EF and BC, FD and CA, DE and AB are collinear.

It can easily be seen that $D \equiv (0, y', z')$; $E \equiv (x', 0, z')$; $F \equiv (x', y', 0)$.

Hence the equation to EF will be

$$-\frac{x}{x'}+\frac{y}{y'}+\frac{z}{z'}=0,$$

which can be written

$$\frac{x}{x'}+\frac{y}{y'}+\frac{z}{z'}=\frac{2x}{x'},$$

shewing that EF passes through the intersection of BC and the line

$$\frac{x}{x'}+\frac{y}{y'}+\frac{z}{z'}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hence it will be plain that the intersections BC, EF; CA, FD; AB, DE are collinear and lie on the line (1).

The Polar Line of a Point with respect to a Triangle.

If in a triangle ABC, the lines AO, BO, CO meet the opposite sides in D, E, F respectively; and if EF, BC; FD, CA; and DE, AB intersect in L, M, N respectively, then the line LMN is called the Polar Line of O with respect to the triangle ABC [2].

14. *Pairs of Lines.*

It is often convenient to have one equation to represent a pair of lines. Thus consider the lines

$$l_1x+m_1y+n_1z=0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

$$l_2x+m_2y+n_2z=0 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

If the coordinates of any point lying on either of the lines (1) and (2) be substituted in the expression

$$(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)=0 \quad \ldots\ldots(3),$$

it will make (3) identically vanish; and, conversely, if the coordinates of any point make (3) vanish, then one or other of its factors must vanish. Hence all points lying on (1) or (2) lie on the locus (3), and all points lying on the locus (3) lie on at least one of the lines (1) or (2).

In particular a line through C will be of the form $l_1x + m_1y = 0$ and a pair of lines through C will be represented by the equation

$$(l_1x + m_1y)(l_2x + m_2y) = 0,$$

which on being multiplied out gives an equation of the form

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Conversely an equation of the form (4) represents a pair of straight lines through C.

15. *To prove that if the general equation of the second degree*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots(1)$$

represent two straight lines, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

will vanish and conversely.

Case I.

Suppose first of all that a, b, c do not all vanish. To fix our ideas suppose that c does not vanish. Write the above equation (1) in the form

$$cz^2 + 2(fy + gx)z + (ax^2 + 2hxy + by^2) = 0 \ldots(2).$$

Solving for z we get

$$z = \frac{-(fy + gx) \pm \sqrt{(fy + gx)^2 - c(ax^2 + 2hxy + by^2)}}{c},$$

or as it may be written

$$z = \frac{-(fy + gx) \pm \sqrt{(g^2 - ac)x^2 + 2(fg - ch)xy + (f^2 - bc)y^2}}{c} \ldots(3).$$

Now if the equation (1) represent two straight lines, the equation (3) must give the two straight lines separately in the form

$$z = px + qy,$$

$$z = rx + sy.$$

Hence the expression under the root in (3) must be a perfect square, the condition for which is

$$(fg - ch)^2 = (f^2 - bc)(g^2 - ca),$$

i.e. $$c(abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

Now c is not = zero by hypothesis, hence the other factor vanishes or in its determinantal form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

CASE II.

Suppose next that a, b, c all vanish, giving us

$$2fyz + 2gzx + 2hxy = 0 \ldots\ldots\ldots\ldots (4).$$

If neither f, g nor h vanish, the equation (4) cannot represent two straight lines; for solving for one variable (say z)

$$2z = -\frac{2hxy}{fy + gx},$$

which plainly cannot represent two straight lines if neither f, g nor h vanish.

If one or more of the quantities f, g or h vanish, (4) obviously represents **a pair** of straight lines. Hence in this case if we suppose **(say)** h to vanish in addition to a, b, c, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 0 & g \\ 0 & 0 & f \\ g & f & 0 \end{vmatrix} = 0.$$

Hence if the given equation (1) *represents two straight lines, the determinant*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \textit{ vanishes.}$$

Next to prove the converse. Suppose that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

to prove that the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

represents two straight lines.

As before, suppose that a, b, c do not all vanish, and let us take $c \neq 0$.

Solve for z as before, getting

$$z = \frac{-(fy+gx) \pm \sqrt{(g^2-ac)x^2 + 2(fg-ch)xy + (f^2-bc)y^2}}{c} \quad \ldots(5).$$

Now $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$; (by hypothesis)

$\therefore \quad c(abc + 2fgh - af^2 - bg^2 - ch^2) = 0,$

or as it may be written

$$(fg - ch)^2 = (f^2 - bc)(g^2 - ac).$$

Hence the expression under the root of (5) is a perfect square, and z can be found linearly in terms of x and y giving *two* straight lines, corresponding to the two signs $\pm$ in (5).

If, however, a, b, c simultaneously vanish, then the vanishing of the given determinant takes the form

or as

$$\begin{vmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{vmatrix} = 0,$$

$z = \frac{-(fy}{}$

i.e. $2fgh=0$. Hence f, g, or $h=0$. Let (say) $h=0$ in addition to a, b, c vanishing. Hence the given equation (1) becomes

$$2fyz+2gzx=0, \text{ i.e. } (fy+gx)z=0,$$

which represents a pair of lines. *Thus if the determinant*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

vanishes, the equation (1) *will represent a pair of straight lines.*

16. The following propositions proved in books on Projective Geometry will be required in what follows ;—

Cross Ratio of four Collinear Points. **If P, Q, R, S be four collinear points, their cross ratio, usually denoted by [PQRS] is defined as $\dfrac{PQ \,.\, RS}{PS \,.\, RQ}$, the usual convention as to the signs of the various segments of the given line being adopted.**

The following method of remembering the order of the letters in the above definition of cross ratio will be found convenient.

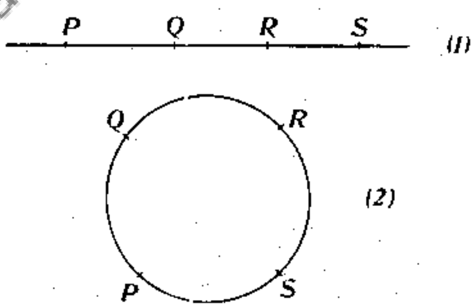

Fig. 6.

Let P, Q, R, S be the four points in this order as in (1) Fig. 6. Imagine them bent round a circle as in (2) Fig. 6. Then for the numerator of the cross ratio, write down the letters in clockwise order and for the denominator write them down in counter-clockwise order, beginning with the same letter in both cases.

Cross Ratio of a Pencil of four Concurrent Lines.

If four lines OA, OB, OC, OD meet in O and if two transversals cut them in the four points P, Q, R, S and P′, Q′, R′, S′ respectively, then will [PQRS] = [P′Q′R′S′], i.e. all transversals are cut by the same pencil of four rays in the same cross ratio [3].

Harmonic Division.

Fig. 6a.

In pure geometry the two points P and Q are said to harmonically separate R and S if

$$\frac{PR}{RQ} + \frac{PS}{SQ} = 0,$$

i.e. if [PRQS] have the particular value − 1.

Also P and Q are called Harmonic Conjugates with respect to R and S [4].

17. *To prove that the two lines*

$$a_1x^2 + 2h_1xy + b_1y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

will harmonically separate the two lines

$$a_2x^2 + 2h_2xy + b_2y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

if

$$a_1b_2 + a_2b_1 = 2h_1h_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Since, as mentioned above, the cross ratio in which any transversal cuts a given set of four concurrent lines is constant we may take any convenient transversal (Fig. 7).

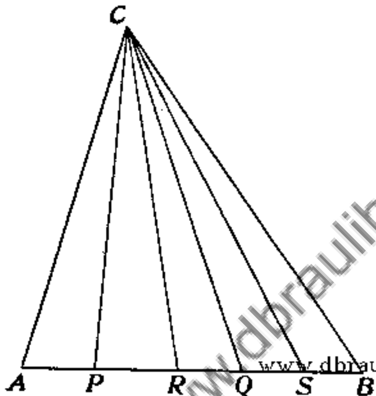

Fig. 7.

Let the given pair of lines (1) cut AB in P and Q.

Let the given pair of lines (2) cut AB in R and S.

Then since it is given that the pair of lines CP and CQ harmonically separate CR and CS we have that the points P and Q harmonically separate R and S.

Let the equations to CP, CQ, CR and CS be $x=py$, $x=qy$, $x=ry$, $x=sy$ respectively.

Since the equation to CP is $x=py$,

$$\therefore \quad P \equiv (p, 1, 0) \quad \text{.................(4).}$$

Hence $\dfrac{AP}{AB} = \dfrac{\Delta APC}{\Delta ABC} =$ actual areal y-coordinate of P

$$= \frac{1}{p+1} \quad \text{.......................(5).}$$

Hence

$$AP = \frac{c_0}{p+1}\,;\ AQ = \frac{c_0}{q+1}\,;\ AR = \frac{c_0}{r+1}\,;\ AS = \frac{c_0}{s+1} \quad (6).$$

Now since P and Q harmonically separate R and S,

$$\therefore \quad \frac{PR}{RQ}+\frac{PS}{SQ}=0,$$

i.e.

$$\frac{AR-AP}{AQ-AR}+\frac{AS-AP}{AQ-AS}=0 \quad \ldots\ldots\ldots\ldots(7).$$

Hence if we substitute from (6) we get

$$\frac{\frac{1}{r+1}-\frac{1}{p+1}}{\frac{1}{q+1}-\frac{1}{r+1}}+\frac{\frac{1}{s+1}-\frac{1}{p+1}}{\frac{1}{q+1}-\frac{1}{s+1}}=0 \quad \ldots\ldots\ldots(8),$$

i.e.

$$2(pq+rs)=(p+q)(r+s) \quad \ldots\ldots\ldots\ldots(9).$$

But since the combined equation to CP and CQ is (1) and their separate equations $x=py$ and $x=qy$,

$$\therefore \quad (x-py)(x-qy)\equiv x^2+2\frac{h_1}{a_1}xy+\frac{b_1}{a_1}y^2.$$

$$\therefore \quad p+q=-2\frac{h_1}{a_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

$$pq=\frac{b_1}{a_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Similarly

$$r+s=-2\frac{h_2}{a_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

$$rs=\frac{b_2}{a_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Substituting from (10), (11), (12), (13) in (9), we get as the required condition

$$a_1b_2+a_2b_1=2h_1h_2.$$

COROLLARY I.

The two lines $ax^2+2hxy+by^2=0$ will harmonically separate CA and BC if $h=0$.

COROLLARY II.

The harmonic conjugate of $y-\lambda x=0$ *with respect to* CA *and* BC *is* $y+\lambda x=0$.

18. <u>The Complete Quadrangle.</u> If P, Q, R, S be four given points and if PQ, RS meet in A; PR, SQ in B; PS, QR in C; the figure so obtained is called the Complete Quadrangle having A, B, C for vertices (5).

Canonical form for the coordinates of any given set of four points.

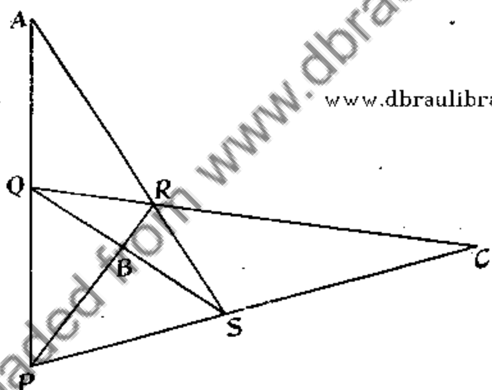

Fig. 8.

Let P, Q, R, S be the four given points (Fig. 8) and take the triangle ABC (as shewn in the diagram) for triangle of reference.

Let $P \equiv (x', y', z')$.

Since Q lies on the line PA,

$$\frac{y}{y'}=\frac{z}{z'},$$

$$\therefore\ Q \equiv (x_0, y', z').$$

Similarly

$$\left.\begin{aligned} Q &\equiv (x_0, y', z') \\ R &\equiv (x', y_0, z') \\ S &\equiv (x', y', z_0) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where x_0, y_0, z_0 are quantities to be determined.

Since

RSA are collinear, $\therefore\ y_0 z_0 = y'z'$ ....(2),

SQB are collinear, $\therefore\ z_0 x_0 = z'x'$ ....(3),

QRC are collinear, $\therefore\ x_0 y_0 = x'y'$ ....(4).

Multiplying (3) and (4) and using (2) we get

$$x_0^2 = x'^2, \quad \therefore\ x_0 = \pm x'.$$

But from (1)

$$x_0 \neq x', \quad \because P \equiv (x', y', z'),$$

and P and Q are not coincident. Hence

$$Q \equiv (-x', y', z').$$

Hence

$$P \equiv (x', y', z'),$$
$$Q \equiv (-x', y', z'),$$
$$R \equiv (x', -y', z'),$$
$$S \equiv (x', y', -z').$$

19. We shall denote lines by small letters p, q, r, s etc. and the point of intersection of the lines p and q will be denoted by pq.

The Complete Quadrilateral. **If p, q, r, s be four given lines, and if a be the line joining pq and rs; b the line joining pr and sq; c the line joining ps and qr; the figure so obtained is called the Complete Quadrilateral and abc is called its Diagonal Triangle**[6].

Choose the triangle abc (or ABC) as triangle of reference (Fig. 9).

Let the equation to p be

$$lx + my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

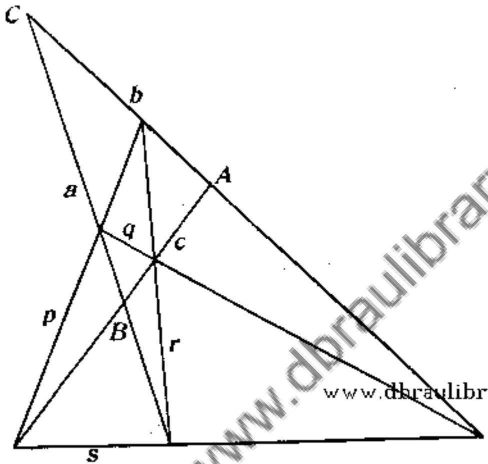

Fig. 9.

Then since q passes through the intersection of p and a its equation will be of the form

$$lx + my + nz = \lambda x$$

or as it may be written

$$l'x + my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

So the equation to r will be

$$lx + m'y + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

So the equation to s will be

$$lx + my + n'z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

r, s, a are concurrent, $\therefore m'n' = mn \quad \ldots\ldots(5).$

s, q, b are concurrent, $\therefore n'l' = nl \quad \ldots\ldots\ldots(6).$

q, r, c are concurrent, $\therefore l'm' = lm \ldots\ldots\ldots(7).$

Multiply (6) and (7) and use (5), $\therefore l'^2 = l^2$.

Hence $l' = \pm l$, but p is

$$lx + my + nz = 0,$$

and since p and q are not coincident we must take $l' = -l$.

$\therefore q$ is $\qquad -lx + my + nz = 0.$

Similarly r is $\qquad lx - my + nz = 0,$

and s is $\qquad lx + my - nz = 0.$

EXAMPLES. I.

1. A point $O \equiv (x', y', z')$ is taken. AO, BO, CO meet the opposite sides of the triangle of reference in A', B', C' respectively. Prove that the equations to $B'C'$, $C'A'$, $A'B'$ are respectively

$$-\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0\,;\ \frac{x}{x'} - \frac{y}{y'} + \frac{z}{z'} = 0\,;\ \frac{x}{x'} + \frac{y}{y'} - \frac{z}{z'} = 0.$$

2. If in question (1) a line through A meets $A'B'$ and $C'A'$ in B'', C'' respectively, shew that the intersection of the lines BB'', CC'' lies on $B'C'$. (Clare etc.)

3. P is a variable point on the fixed line CD passing through C. AP meets BC in Q and BP meets CA in R. Prove that QR passes through a fixed point on AB.

4. The points X, Y are taken on the sides BC, CA respectively of the triangle ABC, so that BX is one third XC and CY is double YA. Shew that AX passes through the middle point of BY. (Sidney Sussex.)

5. D, E, F are the middle points of the sides of a triangle ABC. Any line through F meets BC in P and CA in Q. AP and BQ meet in R. Shew that R lies on CX where CX is parallel to DE.

6. A', B', C' are the middle points of the sides of the triangle ABC and any line is drawn to meet the sides of the triangle $A'B'C'$ in K, L, M. AK, BL, CM meet the sides of ABC in K', L', M' respectively. Prove that $K'L'M'$ is a straight line. (Peterhouse etc.)

7. O is the point (x', y', z') and AO, BO, CO meet the opposite sides of the triangle of reference in D, E, F respectively. The line UVW whose equation is

$$lx + my + nz = 0$$

meets EF in U, FD in V and DE in W. Prove that AU, BV, CW meet the opposite sides of the triangle of reference in three collinear points lying on the line

$$\frac{x}{x'(my' + nz')} + \frac{y}{y'(nz' + lx')} + \frac{z}{z'(lx' + my')} = 0.$$

8. Prove that the actual areal coordinates of the centre of gravity of the three masses m_1, m_2, m_3 at the respective vertices of the triangle of reference are

$$\left(\frac{m_1}{m_1 + m_2 + m_3}, \frac{m_2}{m_1 + m_2 + m_3}, \frac{m_3}{m_1 + m_2 + m_3}\right).$$

9. L, M, N are three points on the sides of the triangle of reference such that AL, BM, CN are concurrent. Through A, B, C lines are drawn parallel to MN, NL, LM respectively. Prove that these lines meet the opposite sides of the triangle of reference respectively in three collinear points.

10. Any three points A_1, B_1, C_1 are taken respectively in the sides BC, CA, AB of the triangle ABC: B_1C_1 and BC intersect in F, C_1A_1 and CA in G, A_1B_1 and AB in H. Also FH and BB_1 intersect in M and FG and CC_1 intersect in N. Prove that MG, NH and BC are concurrent. (Math. Tripos.)

11. Points D, E, F are taken on the sides of a triangle ABC so that AD, BE, CF are concurrent, and L, M, N are the middle points of EF, FD and DE respectively. Prove that AL, BM, CN are concurrent. (Jesus etc.)

12. Prove that if $ABCD$ be a quadrilateral and if x, y, z, u represent the areas of the triangles PAB, PBC, PCD, PDA respectively, when P is a variable point, the areas being taken positive when P is inside the quadrilateral, $xz - yu = 0$ represents the diagonals of the quadrilateral.

13. Within a triangle ABC are taken two points O_1 and O_2. AO_1, BO_1, CO_1 meet the opposite sides A', B', C', and the points of intersection of O_2A, $B'C'$; O_2B, $C'A'$; O_2C, $A'B'$ are respectively D, E, F. Prove that $A'D, B'E, C'F$ will meet in a point which remains the same if O_1, O_2 be interchanged in the construction.

14. Through any point K in the base BC of a triangle ABC, KL is drawn cutting off a triangle KLC, whose area is half that of ABC. If the line drawn from A to the middle point of the base meets KL in O, and RS is drawn through O parallel to AC so as to meet AB, BC in R and S respectively, prove that

$$\frac{OR}{OS} = \frac{CK}{CB}.$$ (Pembroke.)

15. If X, Y, Z be three points lying respectively on the sides BC, CA, AB of a triangle ABC, and if

$$AZ \cdot BX \cdot CY = AY \cdot BZ \cdot CX,$$

then will AX, BY, CZ meet in a point.

16. A straight line meets the sides of the triangle ABC in A', B', C' respectively; the straight line joining A to the intersection of BB' and CC' meets BC in U and V, W are similarly determined. Prove that if any point O be taken, the straight lines joining U, V, W to the intersections of OA, OB, OC with the respective sides of the triangle $A'B'C'$ will pass through a point O' and that OO' will pass through a point whose position is independent of O.

17. Shew that if

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$\equiv (lx + my + nz)(l'x + m'y + n'z),$$

then

$$(mn' - m'n)(gh - af) = (nl' - n'l)(hf - bg)$$
$$= (lm' - l'm)(fg - ch).$$

18. A straight line DE cuts the sides CA, CB of the triangle ABC at D and E respectively. If CXY is drawn cutting DE, AB in X and Y, prove that

$$\frac{CX}{CD} : \frac{CY}{CA} = \frac{EX}{ED} : \frac{BY}{BA}.$$

(Pembroke etc.)

19. From AB, AC, the sides of a triangle ABC, equal parts AF, AE are cut off. FE meets BC in D and the line joining A to the point of intersection of BE and CF meets BC in H; prove that

$$\frac{BD}{DC} = \frac{BH}{HC} = \frac{BF}{CE}.$$

(Sidney Sussex.)

20. ABC is a triangle; through any point P, DPE is drawn parallel to AB cutting CA in D and BC in E; similarly FPG is drawn parallel to CA, F lying on BC and G on AB; and HPK is drawn parallel to BC, H lying on AB and K on AC. DG and EH are produced to intersect in Q. Shew that CPQ is a straight line.

(Corpus etc.)

21. Shew that the centroid of the triangle, the actual areal coordinates of whose vertices are respectively $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, is the point $(x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$.

22. On the sides BC, CA, AB of a triangle ABC, points D, E, F are taken such that

$$BD : DC = CE : EA = AF : FB.$$

Prove that the centroid of the triangle formed by AD, BE, CF coincides with that of the triangle ABC.

(St Catharine's.)

23. ABC is a triangle; AD, BE, CF meet in one point and intersect the opposite sides at D, E, F. EF meets AD in K. Prove that the ratio of the triangles BFD, CED is the same as the ratio of the rectangles contained by BD, FK and CD, EK. (King's etc.)

24. The sides BC, CA, AB of a triangle ABC are divided internally by points A', B', C' so that

$$BA' : A'C = CB' : B'A = AC' : C'B.$$

Also $B'C'$ produced cuts BC externally in A''. Prove that BA'' is to CA'' in the duplicate ratio of CA' to $A'B$.

(Trinity.)

25. $PQRS$ is a quadrangle. U is any point on QR and V any point on PS. On PQ, PR, QS, RS are taken points K, L, M, N respectively so that ULK, UMN are straight lines. If KVM is a straight line, shew that LVN is also a straight line. (Math. Tripos.)

26. The three diagonals of a complete quadrilateral divide one another internally in the ratios l, m, n. Prove that the ratios are connected by an equation of the form

$$nl - mn - lm + 1 = 0,$$

where in each ratio the part of the diagonal is taken as antecedent, which is further from the point of external division of the said diagonal.

27. $PQRS$ is a complete quadrangle where PQ and RS meet in A and PS and QR meet in C. AXY is any line through A and X and Y are any two points on it. PX and QY meet in U while SX and RY meet in V. Shew that UV passes through C.

28. The lines joining the vertices A, B, C of an equilateral triangle to a point P meet the sides opposite A, B, C in A', B', C' respectively; prove that if

$$BA' + CB' + AC' = A'C + B'A + C'B,$$

then P lies on one of the medians of the triangle. (King's.)

29. $ABCD$ is a quadrilateral whose sides BA and CD meet in E, and a line EHK is drawn cutting AD in H and BC in K. If A_1, A_2, A_3, B_1, B_2, B_3 be the middle points of BD, BH, DK, AC, AK, HC respectively, prove that A_2A_3 and B_2B_3 meet on the line joining the middle points of AB and CD. (St Catharine's.)

30. Prove that if ABC, DEF be two coplanar triangles, and if S be a point such that SD, SE, SF cut the sides BC, CA, AB respectively in three collinear points, then SA, SB, SC cut the sides EF, FD, DE respectively in three collinear points. (Trinity.)

31. $PQRS$ is a quadrangle, PQ and RS meeting in A and PS and QR meeting in C. Through A and C are drawn any two lines meeting in O. K is any point on CO. KP meets AO in L, LS meets CO in M, MR meets AO in N. Prove that KQN are collinear.

32. ABC is a triangle and $A'B'C'$ is another inscribed in ABC, so that A' lies on BC, B' on CA, C' on AB. Prove that an infinite number of triangles PQR can be inscribed in $A'B'C'$ and circumscribed to ABC so that P lies on $B'C'$, Q on $C'A'$, R on $A'B'$ and so that QR passes through A, RP through B and PQ through C.

33. $PQRS$ is a complete quadrangle. X is any point on PR and Y any point on QS. QX and PY meet in U while SX and RY meet in V. Prove that UV passes through the intersection of PS and QR.

34. If ABC is a triangle and P a point in BC such that BP is twice PC and if Q is the middle point of AP, and BQ produced cuts AC in R, then BQ is five times QR. (Corpus etc.)

35. Through a fixed point D inside a given triangle ABC a straight line PDQ is drawn cutting AB, AC in P, Q respectively; E being the middle point of AD, PE is joined and produced to meet AC in R; and RM is drawn parallel to AD cutting PQ in M; prove that the locus of M is a straight line.

CHAPTER II

THE CONIC

1. We shall take the general homogeneous equation of the second degree in x, y, z in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots(1).$$

Since every straight line $lx + my + nz = 0$ will meet the above locus in two points (as may be seen by eliminating z and getting a quadratic in $x : y$), (1) will represent a conic.

2. *To find the equation to the tangent at* (x_1, y_1, z_1) *to this conic*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots(1).$$

Let $P \equiv (x_1, y_1, z_1)$ be a point on the conic and let $Q \equiv (x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1)$ be a point lying very near to P, where δx_1 etc. have the meanings usually assigned to them in the Differential Calculus. We wish to find the equation to the line passing through the points P and Q.

Since $P \equiv (x_1, y_1, z_1)$ lies on the conic (1),

$$\therefore ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \ldots(2).$$

Since $Q \equiv (x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1)$ lies on the conic (1),

$$a(x_1 + \delta x_1)^2 + b(y_1 + \delta y_1)^2 + c(z_1 + \delta z_1)^2 + 2f(y_1 + \delta y_1)(z_1 + \delta z_1)$$
$$+ 2g(z_1 + \delta z_1)(x_1 + \delta x_1) + 2h(x_1 + \delta x_1)(y_1 + \delta y_1) = 0.$$

Expanding this, using (2) and neglecting infinitesimals of the second order, e.g. $(\delta x_1)^2$; $\delta y_1 \delta z_1$; etc., we get

$$(ax_1 + hy_1 + gz_1)\,\delta x_1 + (hx_1 + by_1 + fz_1)\,\delta y_1 + (gx_1 + fy_1 + cz_1)\,\delta z_1 = 0 \ldots (3).$$

Now we may write (2) in the form

$$(ax_1 + hy_1 + gz_1)\,x_1 + (hx_1 + by_1 + fz_1)\,y_1 + (gx_1 + fy_1 + cz_1)\,z_1 = 0 \ldots (4).$$

Hence adding (3) and (4) we get

$$(ax_1 + hy_1 + gz_1)(x_1 + \delta x_1) + (hx_1 + by_1 + fz_1)(y_1 + \delta y_1) + (gx_1 + fy_1 + cz_1)(z_1 + \delta z_1) = 0 \ldots (5).$$

Now consider the line

$$(ax_1 + hy_1 + gz_1)\,x + (hx_1 + by_1 + fz_1)\,y + (gx_1 + fy_1 + cz_1)\,z = 0 \ldots (6).$$

By equation (4) the line (6) passes through the point

$$P \equiv (x_1,\ y_1,\ z_1).$$

By equation (5) the line (6) passes through the point

$$Q \equiv (x_1 + \delta x_1,\ y_1 + \delta y_1,\ z_1 + \delta z_1).$$

Now the line that passes through the two adjacent points P and Q is the tangent at P. Hence (6) represents the tangent at P.

3. *Joachimsthal's Ratio Equation.*

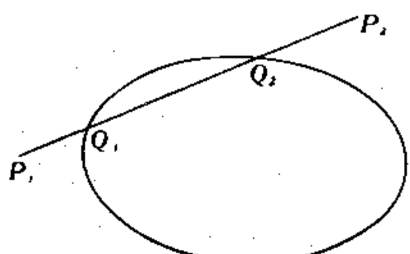

Fig. 10.

Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be two given points denoted by "actual areal coordinates"; let them be joined (Fig. 10) and let P_1P_2 cut the conic in Q_1 and Q_2; *to find the ratios in which the points of section by the conic divide the given line* P_1P_2, *i.e. to find* $\frac{P_1Q_1}{Q_1P_2}$ *and* $\frac{P_1Q_2}{Q_2P_2}$.

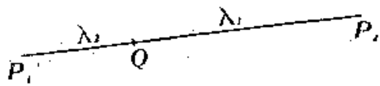

Fig. 11.

Let the point Q divide the line P_1P_2 (Fig. 11) so that

$$\frac{P_1Q}{QP_2} = \frac{\lambda_2}{\lambda_1}.$$

Then by Chap. I, Art. 6,

$$Q \equiv (\lambda_1 x_1 + \lambda_2 x_2,\ \lambda_1 y_1 + \lambda_2 y_2,\ \lambda_1 z_1 + \lambda_2 z_2).$$

Substituting in the equation of the conic we get a quadratic giving the ratios in which the conic divides the line P_1P_2.

Thus

$$a(\lambda_1 x_1 + \lambda_2 x_2)^2 + b(\lambda_1 y_1 + \lambda_2 y_2)^2 + c(\lambda_1 z_1 + \lambda_2 z_2)^2$$
$$+ 2f(\lambda_1 y_1 + \lambda_2 y_2)(\lambda_1 z_1 + \lambda_2 z_2) + 2g(\lambda_1 z_1 + \lambda_2 z_2)(\lambda_1 x_1 + \lambda_2 x_2)$$
$$+ 2h(\lambda_1 x_1 + \lambda_2 x_2)(\lambda_1 y_1 + \lambda_2 y_2) = 0 \ldots (1).$$

Hence collecting the coefficients of λ_1^2, $\lambda_1\lambda_2$, λ_2^2 we get

$$(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1)\lambda_1^2$$
$$+ 2\{ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1)$$
$$+ h(x_1y_2 + x_2y_1)\}\lambda_1\lambda_2$$
$$+ (ax_2^2 + by_2^2 + cz_2^2 + 2fy_2z_2 + 2gz_2x_2 + 2hx_2y_2)\lambda_2^2 = 0 \ldots (2);$$

(2) is called *Joachimsthal's Ratio Equation.*

Case I.

To find the equation to the tangent at any point $P_1 \equiv (x_1, y_1, z_1)$.

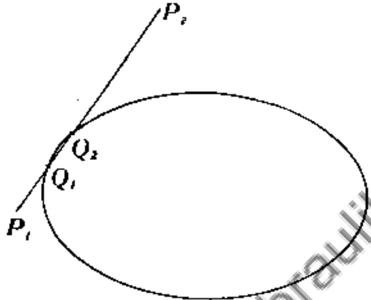

Fig. 12.

Let P_1 be taken very near the curve but not on it (Fig. 12), and let P_1P_2 cut the curve at two points Q_1 and Q_2 so that P_1, Q_1, Q_2 are all very close together and ultimately become coincident.

Then the ratios $\frac{P_1Q_1}{Q_1P_2}$ and $\frac{P_1Q_2}{Q_2P_2}$ ultimately vanish. Hence the two values of $\frac{\lambda_2}{\lambda_1}$ in equation (2) both vanish, giving

$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \ldots (3)$$

and

$$ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1) = 0 \ldots (4).$$

The equation (3) gives the condition that (x_1, y_1, z_1) lies on the curve and (4) gives the condition that (x_2, y_2, z_2) lies on the tangent at (x_1, y_1, z_1). Changing (x_2, y_2, z_2)

to current coordinates we see that the equation to the tangent at (x_1, y_1, z_1) is

$$axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1)$$
$$+ g(zx_1 + xz_1) + h(xy_1 + yx_1) = 0 \ldots (5),$$

or as it may be written

$$x\frac{\partial F}{\partial x_1} + y\frac{\partial F}{\partial y_1} + z\frac{\partial F}{\partial z_1} = 0,$$

or

$$x_1\frac{\partial F}{\partial x} + y_1\frac{\partial F}{\partial y} + z_1\frac{\partial F}{\partial z} = 0,$$

where $F(x, y, z) = 0$ is the equation to the conic.

CASE II.

The Polar Line of a Point with respect to a Conic. **If P_1 be a fixed point and P_2 a variable point such that P_1 and P_2 are harmonically conjugate with respect to the two points in which the line P_1P_2 cuts the conic, then the locus traced out by P_2 is a straight line called the Polar Line of P_1[7].**

Conjugate Points with respect to a Conic. Two points are said to be conjugate with respect to a conic when the straight line joining them is harmonically divided by the conic[8].

To find the equation to the Polar Line of P_1.

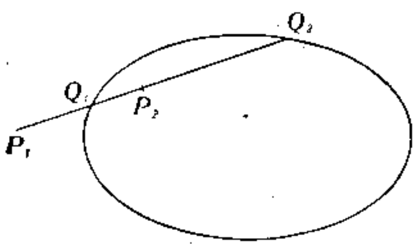

Fig. 13.

The condition that the two points P_1 and P_2 be harmonically conjugate with respect to Q_1 and Q_2 (Fig. 13) is

$$\frac{P_1Q_1}{Q_1P_2} + \frac{P_1Q_2}{Q_2P_2} = 0.$$

(Chap. I, Art. 16.)

Hence the two values for $\frac{\lambda_2}{\lambda_1}$ in (2) must be equal and opposite in sign; whence the coefficient of $\lambda_1\lambda_2$ in (2) must vanish.

The condition that P_1 and P_2 be Conjugate Points with respect to the given conic is, therefore,

$$ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1) = 0.$$

Changing (x_2, y_2, z_2) to (x, y, z) we get that the Polar Line of P_1 is

$$axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) + h(xy_1 + yx_1) = 0,$$

i.e.
$$x\frac{\partial F}{\partial x_1} + y\frac{\partial F}{\partial y_1} + z\frac{\partial F}{\partial z_1} = 0.$$

CASE III.

To find the combined equation to the pair of tangents drawn from P_1 to the conic.

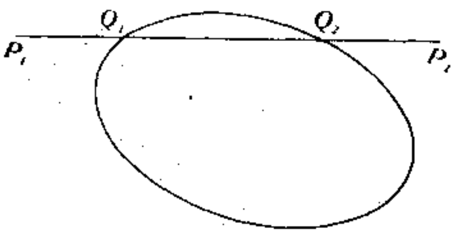

Fig. 14.

Let P_1P_2 cut the conic in two coincident points Q_1, Q_2 (Fig. 14).

In this case the two values for $\frac{\lambda_2}{\lambda_1}$ in the equation (2) become coincident, the condition for which is

$$\{ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1)\}^2 = (ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) \times (ax_2^2 + by_2^2 + cz_2^2 + 2fy_2z_2 + 2gz_2x_2 + 2hx_2y_2).$$

Changing (x_2, y_2, z_2) into (x, y, z) we get, as the equation required to the Pair of Tangents,

$$\{axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) + h(xy_1 + yx_1)\}^2 = (ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) \times (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy).$$

Having now established the equations to the tangent, Polar Line, and Pair of Tangents from a point, it will be plain from what has been said in Chap. I, Art. 6, that we may now dispense with the condition that (x_1, y_1, z_1) and (x_2, y_2, z_2) be "actual areal coordinates." Thus in the above equation to the Pair of Tangents we may replace x_1, y_1, z_1 by $\mu x_1, \mu y_1, \mu z_1$ respectively and x, y, z by $\nu x, \nu y, \nu z$ respectively. μ^2 and ν^2 can be cancelled throughout the resulting equation which will therefore be unaltered in form if we replace the "actual areal coordinates" by areal coordinates.

Similar formulae can be proved with regard to Trilinear Coordinates. Thus the Polar Line of $P_1 \equiv (\alpha_1, \beta_1, \gamma_1)$ will be

$$(a\alpha_1 + h\beta_1 + g\gamma_1)\alpha + (h\alpha_1 + b\beta_1 + f\gamma_1)\beta + (g\alpha_1 + f\beta_1 + c\gamma_1)\gamma = 0.$$

We have only to replace x, y, z in the above formulae by α, β, γ respectively.

4. *To find the equation to a conic circumscribing the triangle of reference.*

Let the conic be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Since $A \equiv (1, 0, 0)$ lies on this conic, $\therefore a = 0$.
So $b = c = 0$.
Hence the form required is

$$2fyz + 2gzx + 2hxy = 0.$$

5. *To find the equation to a conic touching two sides of the triangle of reference, where they are met by the third side.*

Let the conic touch CA and CB at A and B respectively and have as equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots(1).$$

Putting $x = 0$ in (1) to find where this conic cuts BC we ought to get (since the conic meets BC in two coincident points at B) two values for z vanishing, i.e. $z^2 = 0$.

We actually get on putting $x = 0$ in (1)

$$by^2 + cz^2 + 2fyz = 0,$$

$$\therefore \quad b = f = 0.$$

Similarly $\qquad a = g = 0.$

Hence the conic becomes

$$cz^2 + 2hxy = 0.$$

6. Self-conjugate Triangle with respect to a given Conic. A triangle is said to be self-conjugate with respect to a given conic when the polar line of any vertex is the side opposite[9].

To find the equation to a conic, with respect to which the triangle of reference is self-conjugate.

The polar line of $A \equiv (1, 0, 0)$ is

$$ax + hy + gz = 0,$$

and this will be BC, i.e. $x = 0$, if $h = g = 0$.

Similarly $f = 0$.

Hence the conic is of the form

$$ax^2 + by^2 + cz^2 = 0.$$

Common Self-conjugate Triangle with respect to two given Conics. If two conics meet in four distinct points (real or imaginary) P, Q, R, S and if A, B, C be the Vertices of the Complete Quadrangle formed from the four points P, Q, R, S, then ABC is a self-conjugate triangle with respect to each of the two given conics and also with respect to any conic passing through their four points of intersection [10].

Canonical form of the equation to two given conics.

Let the two conics meet in the four points (real or imaginary) P, Q, R, S and let the three vertices of the Complete Quadrangle formed from them (Chap. I, Art. 18) be A, B, C. Choose the triangle ABC as triangle of reference. Then since ABC is self-conjugate with respect to each of the two given conics, their equations will take the form

$$a_1x^2 + b_1y^2 + c_1z^2 = 0,$$

$$a_2x^2 + b_2y^2 + c_2z^2 = 0.$$

7. *To find the equation to a conic which touches the three sides of the triangle of reference.*

Take the equation in the form

$$a^2x^2 + b^2y^2 + c^2z^2 + 2fyz + 2gzx + 2hxy = 0 \quad ...(1).$$

To find where this conic cuts AB, put $z=0$ in (1) getting

$$a^2x^2+b^2y^2+2hxy=0 \quad \text{.............(2).}$$

Now (2) will represent a pair of coincident lines through C

if $\qquad h^2=a^2b^2,$

i.e. if $\qquad h=\pm ab.$

So $\qquad f=\pm bc,$

$\qquad g=\pm ca.$

Hence the general equation to a conic touching the three sides of the triangle of reference will be

$$a^2x^2+b^2y^2+c^2z^2\pm 2bcyz\pm 2cazx\pm 2abxy=0,$$

with the reservation that *we must not take an odd number of the ambiguous signs positive* or else our conic becomes merely a pair of coincident lines,

e.g. $\qquad a^2x^2+b^2y^2+c^2z^2+2bcyz+2cazx+2abxy=0$

is $\qquad (ax+by+cz)^2=0,$

and $\qquad a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx+2abxy=0$

is $\qquad (ax+by-cz)^2=0.$

The equation to an inscribed conic is often very conveniently written in the form

$$\sqrt{ax}+\sqrt{by}+\sqrt{cz}=0.$$

8. Example 1.

The tangents at A, B, C to a circumconic meet the opposite sides respectively in three collinear points.

Let the conic be

$$fyz+gzx+hxy=0 \quad \text{.............(1).}$$

The tangent at A is

$$hy+gz=0,$$

i.e.

$$\frac{y}{g}+\frac{z}{h}=0 \quad \text{.....................(2),}$$

which obviously passes through the intersection of

$$\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0 \quad \text{.....................(3),}$$

and

$$x=0.$$

Hence symmetry shews at once that the tangents at A, B, C will cut the opposite sides respectively in points lying on the line (3).

EXAMPLE 2.

The lines joining the vertices of a triangle to the points of contact of an inscribed conic on the opposite sides respectively are concurrent.

Take the given triangle as triangle of reference and let the conic be

$$a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0.$$

Putting $x=0$ we get, as the equation of the line joining A to the point of contact with BC,

$$by-cz=0.$$

Hence the lines all obviously meet in the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

EXAMPLE 3.

A conic touches the fixed lines CA and CB at A and B respectively. P is a variable point on this conic and the tangent thereat meets BC and CA in Q and R respectively. Prove that the intersection of AQ and BR traces out a conic touching CA and BC at A and B respectively.

Let the conic be

$$cz^2 + 2hxy = 0 \quad \text{.................(1)},$$

and let $P \equiv (x_1, y_1, z_1)$ lie on this conic;

$$\therefore \quad cz_1^2 + 2hx_1y_1 = 0 \quad \text{.................(2)}.$$

The tangent at P will be

$$h(xy_1 + yx_1) + czz_1 = 0 \quad \text{...............(3)}.$$

Putting $x = 0$ in (3) we get, as the equation to AQ,

$$hx_1y + cz_1z = 0 \quad \text{.................(4)}.$$

Similarly the equation to BR is

$$hy_1x + cz_1z = 0 \quad \text{.................(5)}.$$

Eliminating x_1, y_1, z_1 between (2), (4) and (5) we get as the locus required

$$2cz^2 + hxy = 0,$$

which is a conic touching CA and BC at A and B respectively.

9. *To find the pole of a given line with respect to a given conic.*

Let the line be

$$lx + my + nz = 0 \quad \text{.................(1)}.$$

Let its pole be (x', y', z').

Then the polar of this point with respect to the general conic will be

$$(ax' + hy' + gz')x + (hx' + by' + fz')y + (gx' + fy' + cz')z = 0 \quad \text{...(2)}.$$

Identifying (1) and (2) we get

$$\frac{ax' + hy' + gz'}{l} = \frac{hx' + by' + fz'}{m} = \frac{gx' + fy' + cz'}{n},$$

which is a set of linear equations in x', y', z' that we can solve.

10. **The Centre of a Conic. The centre of a given conic is the pole of the Line at Infinity** [11]

To find the centre of a given conic.

Apply the method of last article to find the pole of the Line at Infinity

$$x + y + z = 0.$$

11. *Four-Point System of Conics.*

Consider the equation

$$S_1 + \lambda S_2 = 0 \text{ ....(1)},$$

where S_1 and S_2 each represent the equation to a conic.

First of all we note that the equation (1), being of the second degree in x, y, z, must represent a conic. Also the coordinates of any point common to S_1 and S_2 when substituted in the expressions S_1 and S_2 render each of them equal to zero. Hence the coordinates of any point common to S_1 and S_2 must cause the expression $S_1 + \lambda S_2$ to vanish. Thus all the points common to the conics S_1 and S_2 must lie on the conic $S_1 + \lambda S_2$. Consequently the equation

$$S_1 + \lambda S_2 = 0$$

will be the general equation to all conics passing through the four common points of S_1 and S_2.

We can determine λ so as to make the conic (1) satisfy *one* other condition besides passing through the intersections of S_1 and S_2.

Thus $S_1 + \lambda S_2 = 0$ is the general equation to members of a group of "Four-Point Conics," where S_1 and S_2 are any two members of the system.

If we take the common self-conjugate triangle of the two conics S_1 and S_2 as triangle of reference, members of a four-point system of conics can always be expressed in the form (Chap. II, Art. 6)

$$(a_1x^2+b_1y^2+c_1z^2)+\lambda(a_2x^2+b_2y^2+c_2z^2)=0.$$

COROLLARY.

If $S=0$ be the equation of a conic, $L=0$ and $M=0$ the equations of two straight lines, $S+\lambda LM=0$ will represent a conic passing through the intersections of S and L and of S and M.

12. *Contact of Conics.*

CASE I.

Contact of the first order. If of the four points of intersection of two conics, two become coincident, the conics are said to "touch" or to have "contact of the first order" at the point of coincidence[12].

To find the general equation to a conic S' having contact of the first order with (i.e. touching) a given conic S at a point P.

Let S be the equation to a conic and let another conic S' cut S in the points P, Q, R, S, where P and Q become coincident.

Then by the preceding article the equation to S' must be of the form

$$S+\lambda\widehat{PQ}\,.\,\widehat{RS}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $\widehat{PQ}$ denotes the equation to the line PQ.

Now PQ will in the limit be the tangent at

$$P\equiv(x_1,y_1,z_1);$$

$$\therefore\quad \widehat{PQ}\equiv x\frac{\partial S}{\partial x_1}+y\frac{\partial S}{\partial y_1}+z\frac{\partial S}{\partial z_1}=0 \quad\ldots\ldots(2).$$

Let RS have for its equation

$$lx+my+nz=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence the equation (1) becomes

$$S+\lambda\left(x\frac{\partial S}{\partial x_1}+y\frac{\partial S}{\partial y_1}+z\frac{\partial S}{\partial z_1}\right)(lx+my+nz)=0\ldots(4).$$

CASE II.

Double Contact of two Conics. If of the four points of intersection P, Q, R, S of two conics P, Q and R, S become separately coincident, the conics are said to have "double contact"[13].

To find the general equation to a conic S′ touching a given conic S at two given points.

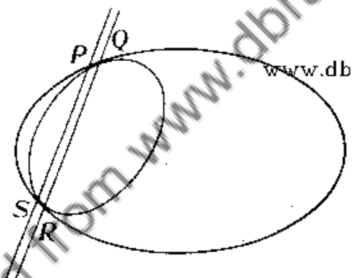

Fig. 15.

Let P, Q and R, S be the two coincident points (Fig. 15). Then it is plain from Case I that the conic S can be exhibited in the form

$$S+\lambda\widehat{PQ}\,.\,\widehat{RS}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

i.e. if $P\equiv(x_1,\ y_1,\ z_1)$ and $R\equiv(x_2,\ y_2,\ z_2)$,

$$S+\lambda\left(x\frac{\partial S}{\partial x_1}+y\frac{\partial S}{\partial y_1}+z\frac{\partial S}{\partial z_1}\right)\left(x\frac{\partial S}{\partial x_2}+y\frac{\partial S}{\partial y_2}+z\frac{\partial S}{\partial z_2}\right)=0\quad(2).$$

The equation to S' may also be taken in the form

$$S+\lambda\widehat{PS}\,.\,\widehat{QR}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let the equation to the two coincident lines PS or QR be

$$lx + my + nz = 0.$$

Then S' will have for its equation

$$S + \lambda (lx + my + nz)^2 = 0 \quad \ldots\ldots\ldots\ldots(4).$$

CASE III.

Contact of the second order. If P, Q, R become coincident, the conics S and S′ are said to have "contact of the second order" at the point P[H].

To find the general equation to a conic S' having contact of the second order with the conic S at the point P (Fig. 16).

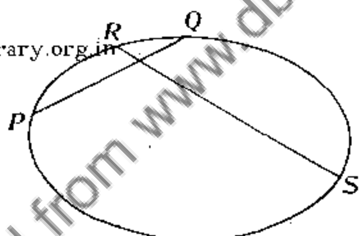

Fig. 16.

S' will now be of the form

$$S + \lambda \widehat{PQ} \,.\, \widehat{RS} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $P \equiv Q \equiv R \equiv (x_1, y_1, z_1)$.

Let the equation to RS be

$$lx + my + nz = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since R lies on (2),

$$\therefore \quad lx_1 + my_1 + nz_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(3).$$

We then get, as the equation to S',

$$S + \lambda \left(x \frac{\partial S}{\partial x_1} + y \frac{\partial S}{\partial y_1} + z \frac{\partial S}{\partial z_1} \right) (lx + my + nz) = 0 \ldots(4).$$

CASE IV.

Contact of the third order, i.e. Osculating Conics. If the four points of intersection become coincident, the conics S and S′ are said to have "contact of the third order" or to "osculate" [15].

To find the general equation to a conic S′ having contact of the third order, i.e. osculating a given conic S at a given point.

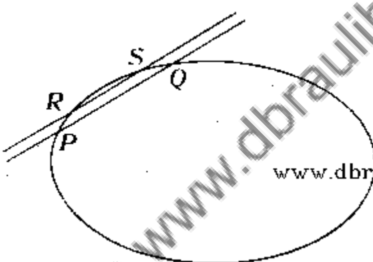

Fig. 17.

In this case the four points P, Q, R, S all become coincident and in the limit PQ and RS become tangents (Fig. 17). Hence S' will have an equation of the form

$$S + \lambda\left(x\frac{\partial S}{\partial x_1} + y\frac{\partial S}{\partial y_1} + z\frac{\partial S}{\partial z_1}\right)^2 = 0.$$

13. EXAMPLE 1.

To every point in a plane there corresponds another point such that these two points are conjugate points with respect to every conic of a given Four-Point System.

Using Chap. II, Art. 11, let the given Four-Point System of Conics be taken in the form

$$a_1x^2 + b_1y^2 + c_1z^2 + \lambda(a_2x^2 + b_2y^2 + c_2z^2) = 0 \ldots(1).$$

Then the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) will be conjugate points with respect to the conic (1) if

$$a_1x_1x_2 + b_1y_1y_2 + c_1z_1z_2 + \lambda(a_2x_1x_2 + b_2y_1y_2 + c_2z_1z_2) = 0 \ldots (2).$$

Now the condition (2) will be satisfied for all values of λ if

$$a_1x_1x_2 + b_1y_1y_2 + c_1z_1z_2 = 0 \quad \ldots\ldots\ldots\ldots (3),$$

$$a_2x_1x_2 + b_2y_1y_2 + c_2z_1z_2 = 0 \quad \ldots\ldots\ldots\ldots (4).$$

Solving for x_2, y_2, z_2 between (3) and (4) we get the coordinates of the point which is conjugate to (x_1, y_1, z_1) with respect to all conics of the given Four-Point System (1).

EXAMPLE 2.

To find the conic which has contact of the second order with

$$fyz + gzx + hxy = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

at C and touches AB at its middle point.

The equation to the tangent at C to the conic (1) is

$$gx + fy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and the general equation to any line through C is

$$lx + my = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence by Chap. II, Art. 12, Case III, the general equation to a conic having contact of the second order with (1) at C will be

$$\lambda(fyz + gzx + hxy) + (gx + fy)(lx + my) = 0 \ldots (4).$$

Now putting $z = 0$ we get that the combined equation to the two lines joining C to the two points in which the conic cuts AB is

$$glx^2 + (fl + gm + \lambda h)xy + fmy^2 = 0 \quad \ldots\ldots (5).$$

Now since the conic (1) is to touch AB at its middle point, the combined equation to the pair of coincident lines joining C to the middle point of AB will be

$$(x-y)^2=0 \quad \text{.....................(6).}$$

Identifying (5) and (6) we get

$$\frac{gl}{1}=\frac{fl+gm+\lambda h}{-2}=\frac{fm}{1} \quad \text{............(7).}$$

Eliminating l, m, λ between (4) and (7) we get as the equation to the required conic

$$(f+g)^2(fyz+gzx+hxy)-h(gx+fy)(fx+gy)=0.$$

14. <u>Ellipse.</u> A conic is called an Ellipse when it intersects the Line at Infinity in two Imaginary Points [16].

<u>Hyperbola.</u> A conic is called a Hyperbola when it intersects the Line at Infinity in two Real Points [17].

<u>Parabola.</u> A conic is called a Parabola when it touches the Line at Infinity [18].

To find whether a conic, whose equation is given in areal coordinates, be an ellipse, a hyperbola or a parabola.

Let the given conic cut the Line at Infinity in the points H and K.

Find the combined equation to the lines CH and CK in the form

$$ux^2+2vxy+wy^2=0 \quad \text{..............(1)}$$

by eliminating z between the equations to the given conic and the Line at Infinity.

Then the given conic will be an Ellipse if the lines CH and CK be imaginary, i.e. if the factors of (1) be imaginary, the condition for which is $v^2<uw$.

We thus have the following scheme;

The condition for an Ellipse is $v^2 < uw$.

The condition for a Hyperbola is $v^2 > uw$.

The condition for a Parabola is $v^2 = uw$.

15. *To find the point where a given parabola touches the Line at Infinity.*

In this case CH and CK of last article coincide and

$$ux^2 + 2vxy + wy^2 = 0$$

will have two coincident factors

$$\sqrt{u}x + \sqrt{w}y = 0 \quad \text{..................(1).}$$

We have therefore only to find the coordinates of the point in which the line (1) cuts the Line at Infinity.

16. Asymptotes. The asymptotes of a given conic are the tangents at the points in which it cuts the Line at Infinity[18].

To find the asymptotes of a given conic.

Find the points in which the Line at Infinity cuts the given conic and then find the equation to the tangents at these points.

17. *To find the combined equation to the asymptotes of a given conic.*

Find the coordinates of the centre by Chap. II, Art. 10, and then write down the combined equation to the tangents drawn from the centre to the conic by Chap. II, Art. 3, Case III.

18. Example 1.

To find the condition that the conic

$$fyz + gzx + hxy = 0 \quad \text{..............(1)}$$

be a parabola.

Let the conic (1) cut the Line at Infinity in the points H and K.

Hence, eliminating z between (1) and the equation to the Line at Infinity

$$x + y + z = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

we get, as the combined equation to CH and CK,

$$(fy + gx)(x + y) - hxy = 0,$$

i.e. $$gx^2 + (f + g - h)xy + fy^2 = 0,$$

which will represent two coincident lines if

$$(f + g - h)^2 = 4fg,$$

i.e. if $$f^2 + g^2 + h^2 - 2gh - 2hf - 2fg = 0,$$

which is therefore the condition that the conic (1) be a parabola.

EXAMPLE 2.

To find the combined equation to the asymptotes of the conic

$$ax^2 + by^2 + cz^2 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let (x_1, y_1, z_1) be the centre of (1).

The polar line of (x_1, y_1, z_1) will be

$$ax_1x + by_1y + cz_1z = 0,$$

and this will be the Line at Infinity if

$$ax_1 = by_1 = cz_1.$$

Thus the centre of the given conic is the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$.

The combined equation to the tangents from this point is, by Chap. II, Art. 3, Case III,

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(ax^2 + by^2 + cz^2) = (x + y + z)^2,$$

which is therefore the equation to the asymptotes.

EXAMPLES. II.

1. A variable conic touches two fixed lines CA and CB at the two fixed points A and B respectively. Prove that the locus of its centre is the median of the triangle ABC bisecting AB.

2. A system of four-point conics is drawn passing through the three vertices of the triangle of reference and the centroid of the triangle. Prove that the locus of the centres of the conics is a conic touching the three sides of the triangle of reference at their middle points.

3. A series of conics is drawn passing through a given point and having a given triangle as a common self-conjugate triangle. Prove that the locus of their centres is a conic circumscribing the given triangle.

4. ABC is a triangle inscribed in a conic. The tangents at B and C meet in A', those at C and A meet in B' and those at A and B meet in C'. Prove that AA', BB', CC' are concurrent.

5. A conic touches the sides of the triangle of reference in D, E, F respectively. Prove that EF meets BC, FD meets CA and DE meets AB in three collinear points.

6. CA and CB are two fixed lines, A and B being also fixed points. O is a fixed point and any line through O meets CA in P and CB in Q. AQ and BP meet in R. Prove that as the line through O varies, R will trace out a conic passing through A, B, C.

7. Prove that in question (6), AB is the polar of O with respect to the conic there obtained.

8. Shew that all conics passing through the system of four points

$$(x', y', z');\ (-x', y', z');\ (x', -y', z');\ (x', y', -z')$$

are self-conjugate with respect to the triangle of reference.

9. If ABC be a triangle and S any conic, prove that the polars of A, B, C with respect to S meet the opposite sides of the triangle, respectively, in three collinear points.

10. If ABC be a triangle and S a given conic, and if A', B', C' be the poles of BC, CA, AB with respect to S, shew that AA', BB', CC' are concurrent.

11. Two conics have a common tangent at C and one touches AB at A and the other touches AB at B. If they meet again in P and Q, prove that CP and CQ harmonically separate CA and CB.

12. Two conics have double contact with one another, the points of contact being A and B. Prove that the polars of any point with respect to the two conics intersect on AB.

13. CA and CB are two fixed tangents to a conic touching it at A and B respectively. O is a variable point on a fixed line. The polar of O with respect to the given conic cuts CA in Q and CB in P. AP and BQ intersect in R. Prove that the locus of R is a conic circumscribing the triangle ABC.

14. A conic is inscribed to a triangle ABC. The polars of A, B, C with respect to the conic cut a fixed line l in L, M, N respectively. AL' is the harmonic conjugate of AL with respect to AB and AC, and BM', CN' are similarly found. Shew that AL', BM', CN' are concurrent.

15. A conic circumscribes a triangle. By a known theorem the tangents at A, B, C meet the opposite sides respectively in three collinear points. If A', B', C' be the poles of BC, CA, AB respectively, then AA', BB', CC' are known to be concurrent. Prove that the aforesaid line of collinearity is the polar line with respect to the triangle of the aforesaid point of concurrency.

16. ABC is a triangle inscribed in a conic. O is a given point. AO meets the conic again in A_1, BO in B_1 and CO in C_1. Prove that B_1C_1, C_1A_1, A_1B_1 meet the sides BC, CA, AB respectively in three collinear points and that the line of collinearity is the polar of O.

17. Two points $(x_1y_1z_1)$, $(x_2y_2z_2)$ will be conjugate with respect to any conic through the four points

$$(x', y', z');\ (-x', y', z');\ (x', -y', z');\ (x', y', -z')$$

if
$$\frac{x_1x_2}{x'^2} = \frac{y_1y_2}{y'^2} = \frac{z_1z_2}{z'^2}.$$

18. The conic

$$cz^2 + 2fyz + 2gzx + 2hxy = 0$$

passes through the vertices A and B of the triangle of reference and cuts CA, CB again in A_1, B_1 respectively. Shew that the equation to A_1B_1 is

$$2gx + 2fy + cz = 0,$$

and that the poles of AB, A_1B_1 and the point C all lie on the line $\dfrac{x}{f} = \dfrac{y}{g}$.

19. Prove that the conics

$$ax^2 + by^2 + cz^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 = 0$$

intersect in the four points given by

$$(\sqrt{bc' - b'c},\ \pm\sqrt{ca' - a'c},\ \pm\sqrt{ab' - a'b}).$$

20. Prove that the conic

$$ax^2 + by^2 + cz^2 = 0$$

will be a parabola if

$$bc + ca + ab = 0.$$

21. Shew that

$$cz^2 + 2hxy = 0$$

will be a parabola if $2c + h = 0$.

22. Prove that

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0$$

will be a parabola if

$$a + b + c = 0,$$

and that its axis will be parallel to the line

$$x(b - c) + y(c - a) + z(a - b) = 0.$$

23. A series of parabolas touch the sides of the triangle of reference in the variable points D, E, F respectively. Prove that the locus of the points of concurrency of AD, BE, CF is a conic passing through A, B, C and having the centroid of the triangle of reference for centre.

24. Prove that the line

$$lx + my + nz = 0$$

will touch the conic

$$ax^2 + by^2 + cz^2 = 0$$

if $$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0,$$

and that the same line will touch

$$cz^2 + 2hxy = 0$$

if $$hn^2 + 2clm = 0.$$

25. If

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

denote a conic, and if A', B', C' be the poles of BC, CA, AB with respect to this conic, prove that A' will be given by the intersections of the two lines

$$\frac{\partial S}{\partial y} = 0, \ \frac{\partial S}{\partial z} = 0,$$

B' by
$$\frac{\partial S}{\partial z} = 0, \ \frac{\partial S}{\partial x} = 0,$$

C' by
$$\frac{\partial S}{\partial x} = 0, \ \frac{\partial S}{\partial y} = 0.$$

26. Shew that the locus of the centre of the conic (areals)

$$lyz + mzx + nxy = 0$$

which passes through (x', y', z') is a conic whose centre is at the point

$$\left(\frac{1 + x'}{4}, \frac{1 + y'}{4}, \frac{1 + z'}{4}\right).$$

(Clare etc.)

27. A conic touches the sides of a triangle ABC, the point of contact of AB being D; prove that the straight line joining the centre of the conic to the middle point of AB will bisect DC. (Clare etc.)

28. If in the conic

$$fyz + gzx + hxy = 0$$

f, g, h are connected by the equation

$$pf + qg + rh = 0,$$

prove that the locus of the centre is a conic. (Corpus etc.)

29. Shew that

$$c(x^2+y^2)+2xy\sqrt{(a-c)(b-c)}-z^2=0$$

has double contact with both of the conics

$$ax^2+by^2-z^2=0,$$

$$bx^2+ay^2-z^2=0.$$

(St John's.)

30. Shew that the family of conics whose equation is

$$(p+\lambda)x^2+(q+\lambda)y^2+(r+\lambda)z^2=0,$$

λ being a variable parameter and the coordinates areal, contains two real parabolas. (King's etc.)

31. Prove that the coordinates of the centre of the conic

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$$

are given by

$$\frac{\alpha}{b_0n+c_0m}=\frac{\beta}{c_0l+a_0n}=\frac{\gamma}{a_0m+b_0l},$$

the coordinates being trilinear. (Downing.)

32. Find the equation of the chord joining the point $(x_1y_1z_1)$ to $(x_2y_2z_2)$ on the conic

$$fyz+gzx+hxy=0$$

in the form

$$\frac{fx}{x_1x_2}+\frac{gy}{y_1y_2}+\frac{hz}{z_1z_2}=0.$$

(Queens'.)

33. Shew that it is possible for a conic to be described round a triangle ABC, such that the tangent at each angle is parallel to the opposite side. (Queens'.)

34. Prove that if two conics have four-point contact at O, and Q is the pole with respect to the second of the tangent at P to the first, OPQ are collinear. (Jesus etc.)

35. Through a fixed point D any conic is drawn having double contact with a given conic. Shew that their common chord intersects the tangent at D to the second conic on a fixed straight line. (Jesus.)

36. The tangents at P, Q, R on a conic form a triangle ABC. Prove that AP, BQ, CR are concurrent and that $P[QARB]$ is a harmonic pencil.

37. The locus of the intersections of the polars of the points of a given line with regard to two given conics is a conic passing through the two poles of the given line and the vertices of the triangle which is self-conjugate with regard to the two conics. What are the tangents at the vertices of the self-conjugate triangle? (Clare etc.)

38. If a conic cut the sides BC, CA, AB of a triangle ABC in the points DD', EE', FF' respectively, then will

$$\frac{BD \, . \, BD'}{CD \, . \, CD'} \times \frac{CE \, . \, CE'}{AE \, . \, AE'} \times \frac{AF \, . \, AF'}{BF \, . \, BF'} = 1.$$

(Carnot's Theorem.)

39. From a given point on a conic any two chords are drawn and through their extremities two chords are drawn parallel to the first two and intersecting the conic again in two other points; prove that the line joining these latter points is parallel to the tangent at the given point. (Peterhouse.)

40. Prove that if an inscribed conic touch the sides of the triangle at D, E, F respectively, and if AD meet the conic again in P etc., the tangents at P, Q, R meet the sides of the triangle respectively in three collinear points.

41. A', B', C' are three points on the sides BC, CA, AB of a triangle such that the lines AA', BB', CC' meet in a point. Shew that a conic can be described to touch the three sides of the triangle at A', B', C' respectively.

42. Two conics have contact of the second order at C and one touches AB at A and the other touches AB at B. If P be the meeting point of the conics other than C, prove that CA and CB harmonically separate the common tangent at C and CP.

43. Shew that

$$\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0$$

and
$$\frac{1}{-lx + my + nz} + \frac{1}{lx - my + nz} + \frac{1}{lx + my - nz} = 0$$

represent the same conic. Explain this fact geometrically. (St John's.)

44. Shew that

$$\frac{l}{x} + \frac{m}{y} + \frac{n}{z} = 0$$

and
$$\sqrt{\frac{y}{m} + \frac{z}{n}} + \sqrt{\frac{z}{n} + \frac{x}{l}} + \sqrt{\frac{x}{l} + \frac{y}{m}} = 0$$

represent the same conic. Explain this fact geometrically. (St John's.)

45. A conic meets the sides BC, CA, AB of a triangle respectively in D, D' and E, E' and F, F'. If the tangents at D, D' respectively meet AB, AC in K, K' and L be the fourth harmonic of B in regard to F, F' and M the fourth harmonic of C in regard to E, E'; prove that $D'L$, DM, KK' are concurrent.

46. If the coordinates are areal, shew that the diameter bisecting all chords of

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0,$$

parallel to

$$lx + my + nz = 0,$$

is $\begin{vmatrix} ux + w'y + v'z & w'x + vy + u'z & v'x + u'y + wz \\ l & m & n \\ 1 & 1 & 1 \end{vmatrix} = 0.$

(Queens'.)

47. Shew that if the lines joining the vertices of the triangle of reference to three of the points in which the conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

meets the opposite sides are concurrent, the same is true of the other three points; and the condition for it is

$$abc - 2fgh - af^2 - bg^2 - ch^2 = 0.$$

(Caius etc.)

48. A conic intersects the sides of the triangle ABC in D, D'; E, E'; F, F' respectively; AD, AD' intersect the conic again in d, d'; BE, BE' in e, e'; CF, CF' in f, f'. Shew that the intersections of dd' with the polar of A; of ee' with the polar of B; of ff' with the polar of C are collinear. (Magdalene.)

49. ABC is a triangle inscribed in a conic; $PLUQ$, $PMVR$ are two chords parallel to BC, CA respectively, L and M lying on AB, U on AC, V on BC. Shew that

$$LU : UQ :: RV : VM.$$

(Math. Trip.)

50. A triangle is inscribed in a conic and through a point in the circumference straight lines are drawn to meet the sides, each being parallel to the diameter conjugate to the side which it meets. Shew that the three points of intersection with the sides are collinear.

51. Two conics have contact of the third order with a given conic in P, Q respectively. Shew that if they touch one another in R, R lies on PQ and that the other points of their intersection will lie on a fixed conic which has double contact with the given conic. (Jesus etc.)

52. Prove that, if the conics $S=0$, $S'=0$ have a pair of common chords $\alpha=0$ and $\beta=0$ such that $S-S'\equiv\alpha\beta$, the equation

$$k^2\alpha^2-2k(S+S')+\beta^2=0$$

represents a conic having double contact with each of the conics S, S'. (Math. Trip.)

53. If a point P traces out a conic passing through two of the vertices of the self-conjugate triangle of a system of four-point conics, then the point of concurrence of the polars of P with respect to the conics of the four-point system traces out another conic passing through the same two vertices of the self-conjugate triangle.

54. Two conics have three-point contact at C and the other common tangent to the two conics touches them respectively at the points A and B. CP and CQ are the respective diameters of the two conics through C. Prove that PQ passes through the intersection of the tangent at C and the common tangent AB.

55. Any point P_1 is taken on the side BC of a triangle ABC, and P_1Q_1, P_1R_1 are drawn parallel respectively to AB, CA meeting CA, AB respectively in Q_1, R_1; Q_1R_2, R_1Q_2 are drawn parallel to BC meeting AB, AC

respectively in R_2, Q_2. Shew that, if lines R_2P_2, Q_2P_2 be drawn respectively parallel to AC, AB, their point of intersection P_2 will be on BC and that the six points P_1, P_2, Q_1, Q_2, R_1, R_2 will lie on a conic. (Sidney Sussex.)

56. Straight lines drawn from A, B, C through a point O meet BC, CA, AB in a, b, c : bc, BC meet in A'; ca, CA in B'; and ab, AB in C': O' is any point on the straight line $A'B'C'$. If AO', BO', CO' meet BC, CA, AB in a', b', c' respectively, prove that a conic can be drawn through a, b, c, a', b', c', O' touching $A'B'C'$ at O'.

(Jesus etc.)

57. If A, B, C is a triangle inscribed within a conic, then an infinite number of self-conjugate triangles PQR can be described in such a way that P lies on BC, Q on CA and R on AB. Also prove that AP, BQ, CR meet in a point and state the locus of this point. (Math. Trip.)

58. Tangents are drawn from a fixed point to each member of a system of conics through four fixed points. Prove that the locus of the points of contact is a cubic curve. (Trinity.)

CHAPTER III

TANGENTIAL COORDINATES

1. *Tangential Coordinates.*

The coefficients l, m, n in the equation to the line

$$lx + my + nz = 0$$

are called the "tangential coordinates" or the "line coordinates" of the line.

Just as we use capital letters to denote points, so shall we use small letters to denote lines, and we shall indicate that the line p has as its tangential coordinates l, m, n thus, $p \equiv (l, m, n)$ on the same analogy as $P \equiv (x, y, z)$ in point coordinates.

N.B. In this chapter we shall assume the point coordinates to be areals unless otherwise stated. l, m, n will then be called the "areal tangential coordinates" or the "areal line coordinates" of the given line.

2. *To find the tangential coordinates of a line passing through the intersection of the lines* (l_1, m_1, n_1); (l_2, m_2, n_2).

The equation to the first line is

$$l_1x + m_1y + n_1z = 0 \quad \text{.................(1).}$$

The equation to the second line is

$$l_2x + m_2y + n_2z = 0 \quad \text{.................(2),}$$

and the equation to any line passing through their intersection is of the form (Chap. I, Art. 8)

$$l_1x + m_1y + n_1z + \lambda(l_2x + m_2y + n_2z) = 0,$$

i.e. $$(l_1 + \lambda l_2)x + (m_1 + \lambda m_2)y + (n_1 + \lambda n_2)z = 0 \ldots (3).$$

Hence its tangential coordinates are

$$(l_1 + \lambda l_2,\ m_1 + \lambda m_2,\ n_1 + \lambda n_2).$$

Compare with this Chap. I, Art. 6, where it is proved that the coordinates of any point on the line joining (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *can be written in the form*

$$(x_1 + \lambda x_2,\ y_1 + \lambda y_2,\ z_1 + \lambda z_2)$$

if we consider ratios only.

3. *To find the tangential coordinates of the line joining two given points.*

Let the points be (x_1, y_1, z_1) and (x_2, y_2, z_2).

Then the equation of the line joining them will be

$$x(y_1z_2 - y_2z_1) + y(z_1x_2 - z_2x_1) + z(x_1y_2 - x_2y_1) = 0,$$

and hence its tangential coordinates will be

$$(y_1z_2 - y_2z_1,\ z_1x_2 - z_2x_1,\ x_1y_2 - x_2y_1).$$

Compare with this result the fact that the point of intersection of the two lines (l_1, m_1, n_1) *and* (l_2, m_2, n_2) *has as its point coordinates* $(m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1)$.

4. *To find the tangential coordinates of the sides of the triangle of reference.*

The side a or BC has for its equation $x = 0$,

i.e. $$1 \cdot x + 0 \cdot y + 0 \cdot z = 0.$$

$\therefore$ in tangential coordinates

$$a, \text{ i.e. } BC \equiv (1, 0, 0),$$
$$b, \text{ i.e. } CA \equiv (0, 1, 0),$$
$$c, \text{ i.e. } AB \equiv (0, 0, 1).$$

5. *To find the areal tangential coordinates of the Line at Infinity.*

Its equation in areal coordinates is

$$x + y + z = 0.$$

$\therefore$ Line at Infinity $\equiv (1, 1, 1)$ in areal tangential coordinates.

COROLLARY.

Line at Infinity $\equiv (a_0, b_0, c_0)$ in trilinear tangential coordinates.

6. *Tangential Equations.*

In point coordinates, if we have an equation

$$F(x, y, z) = 0$$

and if we trace on paper all the points satisfying this equation, we shall get a curve. Now if we have in tangential coordinates a relation $\Phi(l, m, n) = 0$ and if we draw on paper all the lines whose line coordinates satisfy this relation, we shall get an *Envelope*, i.e. a curve considered as touched by all the lines satisfying

$$\Phi(l, m, n) = 0.$$

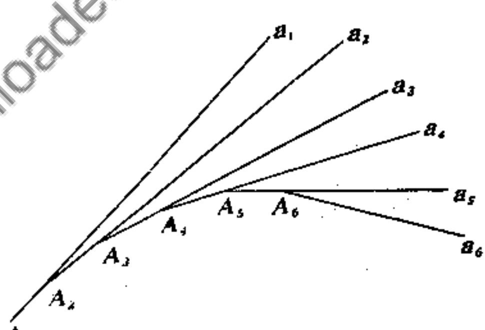

Fig. 18.

In Fig. 18 let a_1, a_2, a_3, a_4, a_5, a_6 be six consecutive lines whose tangential coordinates satisfy the condition

$$\Phi(l, m, n) = 0.$$

Then it is plain that, in the limit, their points of intersection A_1, A_2, A_3, A_4, A_5, A_6 will all lie on a curve and A_1A_2, etc. will be very small arcs of the curve.

Hence just as we draw a curve, say on squared paper, from its point equation, so can we by ruling off lines whose line coordinates satisfy any given tangential equation trace out to any required degree of accuracy the curve which they envelope.

To actually draw any given line (2, 3, 4) on paper we have only to note that its point equation is

$$2x + 3y + 4z = 0,$$

and that this line cuts the sides BC and CA of the triangle of reference in the points $(0, 4, -3)$ and $(2, 0, -1)$ respectively, which can then be joined giving the line required.

7. *To find the tangential equation to a point.*

We wish to find the relationship connecting the tangential coordinates of all lines passing through a given point.

Let the point be (x_1, y_1, z_1).

Then the line (l, m, n), i.e. the line whose point equation is

$$lx + my + nz = 0,$$

will pass through this point if

$$lx_1 + my_1 + nz_1 = 0,$$

which is therefore the tangential equation to the given point.

COROLLARY I.

An equation of the first degree in l, m, n represents the tangential equation to a point.

COROLLARY II.

The coefficients of l, m, n respectively, in the tangential equation to a point, are proportional to the coordinates of the point.

e.g. the point represented by the tangential equation

$$5l - 6m + 7n = 0$$

is the point $(5, -6, 7)$.

COROLLARY III.

The tangential equations of the vertices of the triangle of reference are respectively

for A $\qquad l = 0,$

for B $\qquad m = 0,$

for C $\qquad n = 0.$

8. *Generating Elements.*

In point equations a *point* is to be regarded as the generating element and a line is to be regarded as a *locus of points* (Fig. 19).

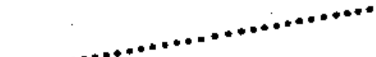

Fig. 19.

In tangential equations a *line* is to be regarded as the generating element and a point is to be regarded as an ***envelope** of lines* (Fig. 20).

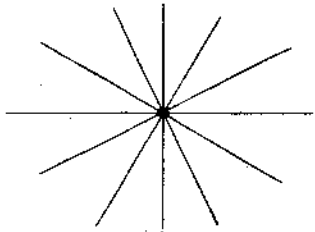

Fig. 20.

9. *Dualistic interpretation of the equation*

$$lx + my + nz = 0.$$

In point coordinates, the equation

$$lx + my + nz = 0$$

is to be regarded as the condition that the variable point (x, y, z) lie on a certain fixed straight line.

In line or tangential coordinates, the equation

$$lx + my + nz = 0$$

is to be regarded as the condition that the variable line (l, m, n) pass through a certain fixed point.

10. EXAMPLE 1.

To find the tangential equation to the centroid of the triangle of reference.

The Centroid $\equiv (1, 1, 1)$ will lie on the line (l, m, n) if

$$l + m + n = 0,$$

which is therefore the tangential equation required.

EXAMPLE 2.

ABC is a triangle and CD is a fixed line through C. P is a variable point on CD and AP meets CB in Q, while BP meets CA in R. Prove that QR always passes through a fixed point on AB.

Consider the line QR as

$$lx + my + nz = 0.$$

$\therefore$ AQ is $$my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and BR is $$lx + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let CD be $$px + qy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The lines (1), (2), (3) will be concurrent if

$$ql + pm = 0.$$

This is a linear equation in l, m, n which may be written

$$q \cdot l + p \cdot m + 0 \cdot n = 0 \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Wherefore (4) is the tangential equation of the point $(q, p, 0)$ which is a fixed point on AB.

Hence $QR \equiv (l, m, n)$ passes through a fixed point on AB.

EXAMPLE 3.

If a fixed conic circumscribe the triangle ABC and if CP and CQ be two variable chords of the conic harmonically separating CA and CB, to prove that the chord PQ passes through a fixed point.

Let the conic be

$$fyz + gzx + hxy = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

and let the chord PQ be

$$lx + my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating z between (1) and (2) we get, as the combined equation to CP and CQ,

$$(lx + my)(gx + fy) = nhxy \quad \ldots\ldots\ldots\ldots(3).$$

The pair of lines (3) will harmonically separate CA

and CB if the coefficient of xy vanish (Chap. I, Art. 17), i.e. if

$$fl + gm - hn = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Hence the chord PQ always passes through the fixed point $(f, g, -h)$.

11. *To find the tangential equation to a conic.*

We wish to find the condition that the line (l, m, n) should touch the conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots(1).$$

Let (x_1, y_1, z_1) be a point on this conic.

Then the tangent thereat will be

$$(ax_1 + hy_1 + gz_1)x + (hx_1 + by_1 + fz_1)y + (gx_1 + fy_1 + cz_1)z = 0 \ldots(2).$$

The tangential coordinates of the line (2) will be proportional to l, m, n if

$$ax_1 + hy_1 + gz_1 = \lambda l \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$hx_1 + by_1 + fz_1 = \lambda m \quad \ldots\ldots\ldots\ldots\ldots(4),$$

$$gx_1 + fy_1 + cz_1 = \lambda n \quad \ldots\ldots\ldots\ldots\ldots(5).$$

Also $$lx_1 + my_1 + nz_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

since (x_1, y_1, z_1) lies on the line (l, m, n).

Eliminating x_1, y_1, z_1, λ between (3), (4), (5), (6) we get, as the tangential equation to the conic,

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0,$$

which on expansion becomes

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

where

$$A = bc - f^2,$$
$$B = ca - g^2,$$
$$C = ab - h^2,$$
$$F = gh - af,$$
$$G = hf - bg,$$
$$H = fg - ch.$$

The following easy rule can be given for writing down A, B, C, F, G, H at once. *Write down the determinant*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

To find the value of any Capital Letter, omit the row and column containing the corresponding small letter and retain the minor determinant thus obtained, prefixing a positive sign if the small letter corresponding to the large letter occur in the centre or one of the corners of the above determinant and a negative sign if anywhere else.

COROLLARY.

The tangential equation to a conic is of the second degree in l, m, n.

12. *To find the tangential coordinates of the two tangents to a given conic passing through a given point.*

Let the conic be

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \ldots (1).$$

Then the line coordinates of any tangent to the above conic passing through the point (x_1, y_1, z_1) must satisfy, in addition to the equation (1), the equation

$$lx_1 + my_1 + nz_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We have therefore to solve for $l : m : n$ between the equations (1) and (2).

13. *Dualistic aspect of a conic.*

In the case of a conic regarded as a locus of points, we must regard *two* of the points of the locus as belonging also to a given straight line (Fig. 21).

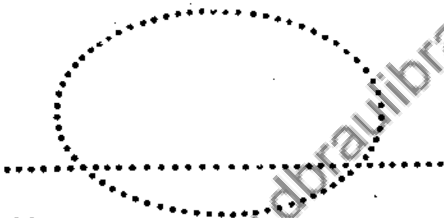

Fig. 21.

In the case of a conic regarded as an envelope of lines, we must regard two of the lines of the envelope as passing through a given point (Fig. 22).

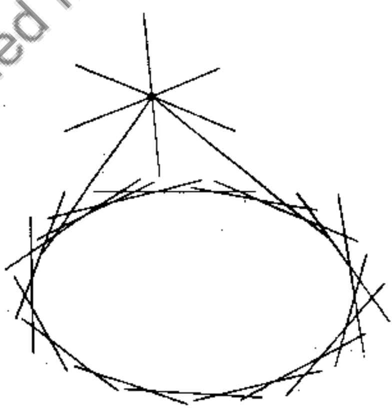

Fig. 22.

14. EXAMPLE 1.

A given conic touches two lines CA and CB at A and B respectively. P is a variable point on the conic. AP meets CB in Q and BP meets CA in R. Prove that QR envelopes a conic.

Taking ABC as triangle of reference, let the conic be

$$z^2 + 2kxy = 0 \quad\text{.....................(1).}$$

Let $QR \equiv (l, m, n)$.

Hence AQ is

$$my + nz = 0 \quad\text{.....................(2),}$$

and BR is

$$lx + nz = 0 \quad\text{.....................(3).}$$

Hence the intersection P of (2) and (3) is

$$\left(\frac{1}{l}, \quad \frac{1}{m}, \quad -\frac{1}{n}\right),$$

and since P lies on the conic (1) we get

$$2kn^2 + lm = 0,$$

which is the tangential equation to a conic.

EXAMPLE 2.

To find the condition that the conic

$$fyz + gzx + hxy = 0 \quad\text{..............(1)}$$

be a parabola.

The tangential equation to this conic is, by Chap. III, Art. 11,

$$f^2l^2 + g^2m^2 + h^2n^2 - 2ghmn - 2hfnl - 2fglm = 0.$$

The Line at Infinity whose line coordinates are (1, 1, 1) will touch this conic if

$$f^2 + g^2 + h^2 - 2gh - 2hf - 2fg = 0,$$

which is therefore the condition required.

15. *To find the tangential equation of the point of contact of the tangent* (l_1, m_1, n_1).

Let the conic be

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \ldots(1),$$

or, in point coordinates,

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots(2).$$

Let the point of contact of the tangent (l_1, m_1, n_1) be

$$(x_1, y_1, z_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The tangent to the conic at (x_1, y_1, z_1) is

$$(ax_1 + hy_1 + gz_1)\,x + (hx_1 + by_1 + fz_1)\,y + (gx_1 + fy_1 + cz_1)\,z = 0 \ldots(4).$$

Hence from (3) and (4)

$$ax_1 + hy_1 + gz_1 = \lambda l_1,$$
$$hx_1 + by_1 + fz_1 = \lambda m_1,$$
$$gx_1 + fy_1 + cz_1 = \lambda n_1.$$

$\therefore$ solving for x_1, y_1, z_1 we get

$$x_1 = \lambda \frac{\begin{vmatrix} l_1 & h & g \\ m_1 & b & f \\ n_1 & f & c \end{vmatrix}}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}} = \lambda \frac{Al_1 + Hm_1 + Gn_1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}} \ldots(5),$$

and so for y_1 and z_1.

Now the tangential equation to the point (x_1, y_1, z_1) is

$$lx_1 + my_1 + nz_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Hence from (5) and (6) the tangential equation to the point of contact (x_1, y_1, z_1) of the tangent (l_1, m_1, n_1) to the given conic is

$$l(Al_1 + Hm_1 + Gn_1) + m(Hl_1 + Bm_1 + Fn_1) + n(Gl_1 + Fm_1 + Cn_1) = 0,$$

or as it may be written

$$l\frac{\partial\Sigma}{\partial l_1}+m\frac{\partial\Sigma}{\partial m_1}+n\frac{\partial\Sigma}{\partial n_1}=0,$$

or
$$l_1\frac{\partial\Sigma}{\partial l}+m_1\frac{\partial\Sigma}{\partial m}+n_1\frac{\partial\Sigma}{\partial n}=0,$$

which is of the same form as the equation to the tangent in areals or trilinears (Chap. II, Art. 3).

16. *To find the tangential equation to the pole of the line (l', m', n').*

Let (l', m', n') meet the conic in the points P_1 and P_2. Then the pole of (l', m', n') is the intersection of the tangent at P_1 with the tangent at P_2, the tangential coordinates of these two tangents being taken to be (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Since (l', m', n') passes through the point of contact of the tangent (l_1, m_1, n_1),

$$\therefore\quad l_1\frac{\partial\Sigma}{\partial l'}+m_1\frac{\partial\Sigma}{\partial m'}+n_1\frac{\partial\Sigma}{\partial n'}=0\quad\ldots\ldots\ldots\ldots(1).$$

So
$$l_2\frac{\partial\Sigma}{\partial l'}+m_2\frac{\partial\Sigma}{\partial m'}+n_2\frac{\partial\Sigma}{\partial n'}=0\quad\ldots\ldots\ldots\ldots(2).$$

Now (1) and (2) are the conditions that (l_1, m_1, n_1) and (l_2, m_2, n_2) each pass through the point whose tangential equation is

$$l\frac{\partial\Sigma}{\partial l'}+m\frac{\partial\Sigma}{\partial m'}+n\frac{\partial\Sigma}{\partial n'}=0\ldots\ldots\ldots\ldots\ldots(3),$$

which will therefore be the tangential equation to the pole of (l', m', n')

N.B. The tangential equation to the pole is of the same form as the point equation to the polar (Chap. II, Art. 3).

17. Conjugate Lines with respect to a given Conic. **Two lines are said to be conjugate with respect to a given conic when they harmonically separate the pair of tangents drawn from their point of intersection to the given conic**[20].

If two lines be conjugate with respect to a given conic, the pole of either with respect to the conic lies on the other[21].

To find the condition that the two lines (l_1, m_1, n_1) *and* (l_2, m_2, n_2) *be conjugate with respect to the conic.*

The tangential equation to the pole of (l_1, m_1, n_1) is, by Chap. III, Art. 16,

$$l\frac{\partial\Sigma}{\partial l_1}+m\frac{\partial\Sigma}{\partial m_1}+n\frac{\partial\Sigma}{\partial n_1}=0,$$

i.e. the pole of the line (l_1, m_1, n_1) is the point

$$\left(\frac{\partial\Sigma}{\partial l_1},\ \frac{\partial\Sigma}{\partial m_1},\ \frac{\partial\Sigma}{\partial n_1}\right)$$

in point coordinates.

This point will lie on the line (l_2, m_2, n_2) if

$$l_2\frac{\partial\Sigma}{\partial l_1}+m_2\frac{\partial\Sigma}{\partial m_1}+n_2\frac{\partial\Sigma}{\partial n_1}=0,$$

which is the condition required and which may also be written

$$l_1\frac{\partial\Sigma}{\partial l_2}+m_1\frac{\partial\Sigma}{\partial m_2}+n_1\frac{\partial\Sigma}{\partial n_2}=0.$$

N.B. This is of the same form as the condition for conjugate points in point coordinates (Chap. II, Art. 3).

18. *To find the tangential equation to a conic touching the sides of the triangle of reference.*

Let the conic be

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

Then the line a, i.e. $BC \equiv (1, 0, 0)$, touches this conic if

$$A = 0.$$

So the line b, i.e. $CA \equiv (0, 1, 0)$, touches this conic if

$$B = 0.$$

So the line c, i.e. $AB \equiv (0, 0, 1)$, touches this conic if

$$C = 0.$$

Hence the equation required is

$$2Fmn + 2Gnl + 2Hlm = 0.$$

N.B. The point equation to a circum-conic is

$$2fyz + 2gzx + 2hxy = 0.$$

19. *To find the tangential equation to a conic touching the two sides of the triangle of reference where they are met by the third side.*

Let BC and CA touch the conic at B and A respectively.

Hence by last article $A = B = 0$.

The conic will therefore take the form

$$Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \quad\text{.........(1).}$$

Now the pole of the line AB whose tangential coordinates are (0, 0, 1) has for its tangential equation (Chap. III, Art. 16)

$$Gl + Fm + Cn = 0 \quad\text{..................(2).}$$

But, by hypothesis, the pole of AB is C whose tangential equation is

$$n = 0 \quad\text{...........................(3).}$$

Identifying the equations (2) and (3) we get

$$F = G = 0.$$

Hence the equation to the given conic becomes

$$Cn^2 + 2Hlm = 0.$$

N.B. The point equation to a conic touching CA at A and BC at B is

$$cz^2 + 2hxy = 0.$$

20. *To find the tangential equation to a conic with respect to which the triangle of reference is self-conjugate.*

The pole of the line BC whose tangential coordinates are (1, 0, 0) has for its tangential equation

$$Al + Hm + Gn = 0 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

But, by hypothesis, the pole of BC is A whose tangential equation is

$$l = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Comparing (1) and (2) we get $G = H = 0$. So $F = 0$. Hence the equation is

$$Al^2 + Bm^2 + Cn^2 = 0.$$

N.B. The point equation to a conic with respect to which the triangle of reference is self-conjugate is

$$ax^2 + by^2 + cz^2 = 0.$$

21. *To find the tangential equation of a conic circumscribing the triangle of reference.*

All lines through A must satisfy the condition $l = 0$, and hence to find the line coordinates of the two tangents drawn from A to the conic

$$A^2l^2 + B^2m^2 + C^2n^2 + 2Fmn + 2Gnl + 2Hlm = 0 \ldots(1)$$

put $l = 0$, getting

$$B^2m^2 + 2Fmn + C^2n^2 = 0 \quad \ldots\ldots\ldots\ldots(2).$$

Now if the conic pass through A the tangents from A to the conic become coincident.

Hence (2) will be a perfect square,

i.e. $$F^2 = B^2C^2.$$

$$\therefore \quad F = \pm BC.$$

Hence the conic (1) becomes

$$A^2l^2 + B^2m^2 + C^2n^2 \pm 2BCmn \pm 2CAnl \pm 2ABlm = 0 \ldots (3),$$

where we must take such values of the ambiguous signs as will not make (3) a perfect square, i.e. we must take an odd number of them negative.

N.B. The point equation to an inscribed conic is

$$a^2x^2 + b^2y^2 + c^2z^2 \pm 2bcyz \pm 2cazx \pm 2abxy = 0.$$

22. *To find the point equation of the conic*

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

Reasoning as in Chap. III, Art. 11, we get as the point equation

$$\begin{vmatrix} A & H & G & x \\ H & B & F & y \\ G & F & C & z \\ x & y & z & 0 \end{vmatrix} = 0,$$

which reduces to the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0;$$

on expanding and cancelling out the factor

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

as is easily seen if we remember that

$$BC - F^2 = \Delta a \text{ etc. and } GH - AF = \Delta f \text{ etc.}$$

23. EXAMPLE 1.

The conic

$$Fmn + Gnl + Hlm = 0$$

touches the sides of the triangle of reference at the points D, E, F respectively. To prove that the lines AD, BE, CF are concurrent and to find the coordinates of their point of intersection.

The tangential equation of D, the point of contact of the line BC whose tangential coordinates are $(1, 0, 0)$, will be

$$Hm + Gn = 0,$$

or as it may be written

$$0 . l + \frac{1}{G} . m + \frac{1}{H} . n = 0 \quad \text{...........(1).}$$

Hence in point coordinates

$$D \equiv \left(0, \frac{1}{G}, \frac{1}{H}\right) \quad \text{..................(2),}$$

and the equation to the line AD will be

$$Gy = Hz \text{.......................(3).}$$

Similarly for BE and CF, whence these lines will all obviously meet in the point

$$\left(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}\right).$$

EXAMPLE 2.

To find the point equation to the pair of lines whose tangential coordinates are given by

$$Al^2 + 2Hlm + Bm^2 = 0 \text{..............(1),}$$

$$n = 0 \text{..............(2).}$$

Let (x, y, z) be a point on either of the given lines. Then any line (l, m, n) passing through this point must satisfy the equation

$$lx + my + nz = 0 \quad \text{.................(3)}.$$

Eliminating l, m, n between (1), (2), (3) we get

$$Ay^2 - 2Hxy + Bx^2 = 0,$$

which is therefore the point equation required.

EXAMPLE 3.

If ABC be a triangle and Σ a conic, and if the tangents from A to Σ meet BC in A′, A″ and so on, to prove that the six points A′, A″, B′, B″, C′, C″ all lie on a conic.

If the conic be

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \text{...(1)},$$

then the tangents from A to the conic will be obtained by putting $l = 0$ in (1) and solving the resulting quadratic in $m : n$, viz.

$$Bm^2 + 2Fmn + Cn^2 = 0 \text{..............(2)}.$$

The pair of lines thus obtained will have as their combined equation in point coordinates

$$\frac{y^2}{B} + \frac{z^2}{C} - \frac{2F}{BC}yz = 0 \quad \text{..............(3)}.$$

Consider now the conic

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} - 2\frac{F}{BC}yz - 2\frac{G}{CA}zx - 2\frac{H}{AB}xy = 0 \text{ ...(4)}.$$

This passes through A', A'' in virtue of (3).

For, putting $x=0$ in (4), we get as the combined equation to the two lines joining A to the points of section of the conic with BC

$$\frac{y^2}{B}+\frac{z^2}{C}-2\frac{F}{BC}yz=0,$$

i.e. AA', AA'' by (3).

The conic (4) will be seen to pass also through B', B'', C', C'' by symmetry.

24. Common Tangents to two Conics. Every pair of Conics has four common tangents [22].

The Diagonal Triangle of the Complete Quadrilateral formed of the four common tangents to two given conics is self-conjugate with respect to each of the two given conics [23]

Canonical Form in Tangential Coordinates for the equation to two given conics.

Draw the four common tangents to the two given conics.

Refer the conics to the Diagonal Triangle of the Complete Quadrilateral, formed by their four common tangents, as triangle of reference.

Since this triangle is self-conjugate with respect to each of the conics, their equations will be of the form

$$A_1l^2+B_1m^2+C_1n^2=0,$$

$$A_2l^2+B_2m^2+C_2n^2=0.$$

25. System of Four-Line Conics. A system of Four-Line Conics is a system of Conics each member of which touches four given straight lines [24].

To find the tangential equation to the system of conics touching the four common tangents of the two given conics

Σ_1 and Σ_2 whose equations are given in tangential coordinates.

Consider the equation

$$\Sigma_1 + \lambda\Sigma_2 = 0.$$

Being of the second degree in l, m, n it represents a conic.

Also since the tangential coordinates of any of the four common tangents to the conics Σ_1 and Σ_2 satisfy both

$$\Sigma_1 = 0 \text{ and } \Sigma_2 = 0,$$

they will also satisfy

$$\Sigma_1 + \lambda\Sigma_2 = 0.$$

Hence $\Sigma_1 + \lambda\Sigma_2 = 0$

is the general tangential equation to all conics touching the four common tangents of Σ_1 and Σ_2.

COROLLARY.

The general tangential equation to all conics touching the four common tangents of two given conics can be taken in the form

$$(A_1l^2 + B_1m^2 + C_1n^2) + \lambda(A_2l^2 + B_2m^2 + C_2n^2) = 0.$$

26. EXAMPLE 1.

Of all the members of a Four-Line System of Conics, one is a parabola.

Take the equation to a conic of the given Four-Line System in the form

$$(A_1l^2 + B_1m^2 + C_1n^2) + \lambda(A_2l^2 + B_2m^2 + C_2n^2) = 0.$$

This will touch the Line at Infinity, whose line coordinates are (1, 1, 1), if

$$(A_1 + B_1 + C_1) + \lambda(A_2 + B_2 + C_2) = 0,$$

which gives one value for λ.

EXAMPLE 2.

The join of the poles of two given lines with respect to all the members of a Four-Line System envelopes a conic.

Let the given lines be (l_1, m_1, n_1) and (l_2, m_2, n_2).

The tangential equations of the poles of these lines with respect to the conic

$$(A_1l^2 + B_1m^2 + C_1n^2) + \lambda(A_2l^2 + B_2m^2 + C_2n^2) = 0 \ldots(1)$$

are respectively

$$(A_1ll_1 + B_1mm_1 + C_1nn_1) + \lambda(A_2ll_1 + B_2mm_1 + C_2nn_1) = 0 \ldots(2),$$

$$(A_1ll_2 + B_1mm_2 + C_1nn_2) + \lambda(A_2ll_2 + B_2mm_2 + C_2nn_2) = 0 \ldots(3).$$

Hence if (l, m, n) be the line satisfying (2) and (3) simultaneously, it will be the line joining the poles of the given lines with respect to that conic of the Four-Line System whose parameter is λ.

Eliminating λ we get, as the envelope required,

$$(A_1l_1l + B_1m_1m + C_1n_1n)(A_2l_2l + B_2m_2m + C_2n_2n) = (A_1l_2l + B_1m_2m + C_1n_2n)(A_2l_1l + B_2m_1m + C_2n_1n).$$

27. *Degenerate Conics in Tangential Coordinates.*

The general equation to a conic in Tangential Coordinates is

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \ldots(1).$$

If, however,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0 \ldots\ldots\ldots\ldots(2),$$

it can be shewn, as in Chap. I, Art. 15, that (1) can be resolved into a pair of linear factors; and since in tangential coordinates linear equations represent *points*, we see that if (2) vanishes, (1) will represent a pair of points.

Thus in tangential coordinates a degenerate conic is a "pair of points," whereas in point coordinates a degenerate conic is a "pair of lines."

28. *If P_1 and P_2; Q_1 and Q_2 be two pairs of points (Fig. 23), to find the general equation to a system of conics touching the four lines P_1Q_1, P_1Q_2, P_2Q_1, P_2Q_2.*

Let

$$P_1 \equiv (x_1, y_1, z_1); \quad P_2 \equiv (x_2, y_2, z_2);$$
$$Q_1 \equiv (x_1', y_1', z_1'); \quad Q_2 \equiv (x_2', y_2', z_2').$$

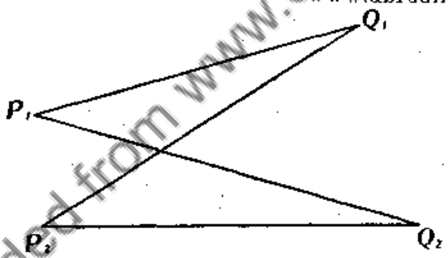

Fig. 23.

Now we may regard P_1; P_2 and Q_1; Q_2 as two degenerate Point Conics having the four lines P_1Q_1, P_1Q_2, P_2Q_1, P_2Q_2 as their four common tangents.

But the tangential equations to the degenerate Point Conics P_1; P_2 and Q_1; Q_2 are respectively (Chap. III, Art. 7)

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2) = 0 \ldots\ldots(1),$$

$$(lx_1' + my_1' + nz_1')(lx_2' + my_2' + nz_2') = 0 \ldots\ldots(2).$$

Hence, applying Chap. III, Art. 25, the general equation to all conics touching the four lines P_1Q_1, P_1Q_2, P_2Q_1, P_2Q_2 will be

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2) + \lambda(lx_1' + my_1' + nz_1')(lx_2' + my_2' + nz_2') = 0.$$

COROLLARY I.

The general tangential equation to all conics touching the four common tangents drawn from $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ to the conic Σ will be

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2) + \lambda\Sigma = 0.$$

COROLLARY II.

To find the tangential equation of a conic Σ' touching another conic Σ at a given point.

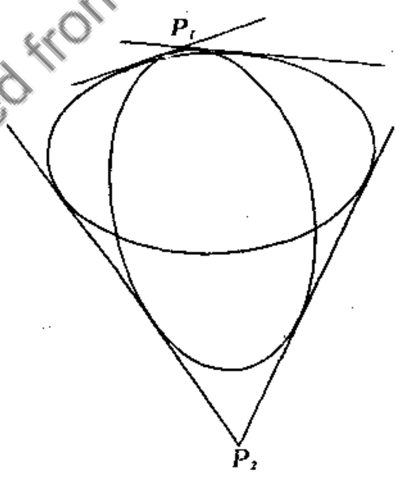

Fig. 24.

Let P_1 be a point very near the curve Σ (Fig. 24). Then the tangents from P_1 to Σ will nearly coincide. Also if a conic Σ' touch the tangents drawn from P_1 to Σ, it will touch these two tangents at points very near to P_1, and in the limit when P_1 lies actually on Σ, Σ and Σ' will touch at P_1.

Let (l_1, m_1, n_1) be the tangent at P_1 to Σ. Then the tangential equation to the point P_1 will be

$$l\frac{\partial\Sigma}{\partial l_1} + m\frac{\partial\Sigma}{\partial m_1} + n\frac{\partial\Sigma}{\partial n_1} = 0.$$

Hence the tangential equation to Σ' will be of the form

$$\left(l\frac{\partial\Sigma}{\partial l_1} + m\frac{\partial\Sigma}{\partial m_1} + n\frac{\partial\Sigma}{\partial n_1}\right)(lx_2 + my_2 + nz_2) + \lambda\Sigma = 0,$$

where P_2 is a point whose coordinates are (x_2, y_2, z_2).

COROLLARY III.

To find the tangential equation of a conic Σ' having double contact with a given conic Σ at two given points P_1 and P_2, the tangents at which are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Obviously

$$\left(l\frac{\partial\Sigma}{\partial l_1} + m\frac{\partial\Sigma}{\partial m_1} + n\frac{\partial\Sigma}{\partial n_1}\right)\left(l\frac{\partial\Sigma}{\partial l_2} + m\frac{\partial\Sigma}{\partial m_2} + n\frac{\partial\Sigma}{\partial n_2}\right) + \lambda\Sigma = 0.$$

COROLLARY IV.

To find the tangential equation of a conic Σ' having double contact with a given conic Σ at the points where it is met by the line (l_1, m_1, n_1).

Let Σ' be a conic touching the four common tangents from P_1 and P_2 to Σ (Fig. 25). Then in the limit when P_1 and P_2 move up close together and ultimately coincide, Σ' will touch Σ at the points where the tangents from P_1 touch Σ, i.e. at the points where the polar of P_1

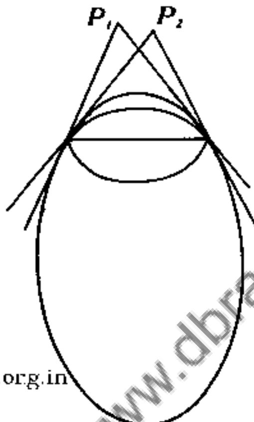

Fig. 25.

cuts Σ. If this polar be (l_1, m_1, n_1), the tangential equation to P_1 will be

$$l\frac{\partial\Sigma}{\partial l_1} + m\frac{\partial\Sigma}{\partial m_1} + n\frac{\partial\Sigma}{\partial n_1} = 0$$

(Chap. III, Art. 16), and Σ' will be

$$\left(l\frac{\partial\Sigma}{\partial l_1} + m\frac{\partial\Sigma}{\partial m_1} + n\frac{\partial\Sigma}{\partial n_1}\right)^2 + \lambda\Sigma = 0.$$

COROLLARY V.

To find the tangential equation of a conic Σ' having contact of the second order with a given conic Σ.

Let the conic Σ' cut Σ in three points P, Q, R very close together (Fig. 26). Let S be the other point. Let us draw the four common tangents of Σ and Σ' and we then see that three tangents very nearly coincide at P. Let P_1 be the point of coincidence of P, Q, R, and let the tangent thereat be (l_1, m_1, n_1) and let $P_2 \equiv (x_2, y_2, z_2)$ (Fig. 27).

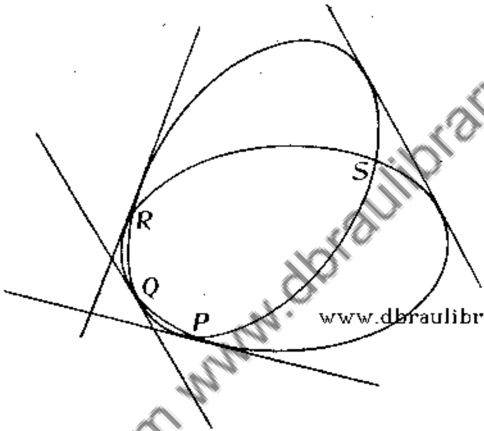

Fig. 26.

be the point of intersection of the fourth common tangent and (l_1, m_1, n_1). Then Σ' becomes

$$\left(l\frac{\partial\Sigma}{\partial l_1} + m\frac{\partial\Sigma}{\partial m_1} + n\frac{\partial\Sigma}{\partial n_1}\right)(lx_2 + my_2 + nz_2) + \lambda\Sigma = 0,$$

where $$l_1x_2 + m_1y_2 + n_1z_2 = 0,$$

since in this case P_2 lies on the tangent at P_1.

COROLLARY VI.

To find the tangential equation to a conic Σ' having contact of the third order at P_1 with Σ.

In this case Σ' cuts Σ in four coincident points and the four common tangents become coincident. Hence P_1 and P_2 of Corollary V become coincident, and we get for Σ'

$$\left(l\frac{\partial\Sigma}{\partial l_1}+m\frac{\partial\Sigma}{\partial m_1}+n\frac{\partial\Sigma}{\partial n_1}\right)^2+\lambda\Sigma=0.$$

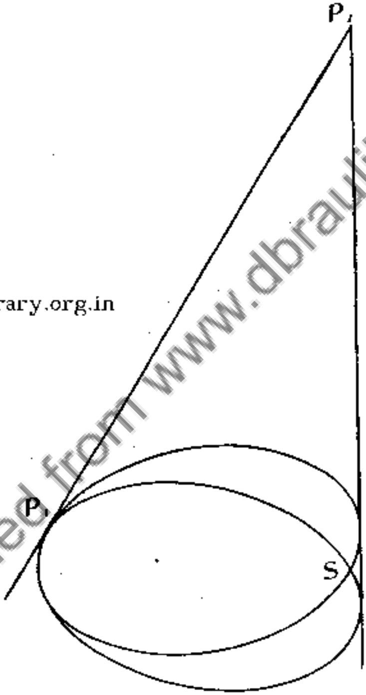

Fig. 27.

29. EXAMPLE 1.

If a series of conics touch each of two given straight lines and have double contact with a given conic, the poles of the chords of contact will lie on one or other of two given straight lines.

Let the conic be

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

and let the two straight lines be CA and CB.

Let (x, y, z) be the pole of the chord of contact. Then the system of variable conics will be

$$(lx + my + nz)^2$$
$$= \lambda (Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm) \ldots (1).$$

Now (1) will touch $CB \equiv (1, 0, 0)$ and $CA \equiv (0, 1, 0)$ if $x^2 = \lambda A$ and $y^2 = \lambda B$ respectively. Hence eliminating λ the poles lie on

$$\frac{x^2}{A} = \frac{y^2}{B},$$

whence the proposition is evident.

EXAMPLE 2.

If a series of conics have contact of the third order with a given conic Σ at a given point C on Σ, the locus of their centres is a line through C.

Let the tangent at C to the fixed conic be chosen as CA.

Hence $\Sigma \equiv Al^2 + Cn^2 + 2Fmn + 2Gnl = 0 \quad \ldots\ldots(1).$

Since the tangential equation to the point C is $n = 0$, the required system of conics will be (Chap. III, Art. 28, Cor. VI)

$$Al^2 + Cn^2 + 2Fmn + 2Gnl + \lambda n^2 = 0 \quad \ldots\ldots(2).$$

The centre, being the pole of the Line at Infinity (1, 1, 1), will have as its tangential equation

$$(A + G)\, l + Fm + (C + F + G + \lambda)\, n = 0,$$

whose point coordinates evidently satisfy the fixed line

$$\frac{x}{A + G} = \frac{y}{F}.$$

30. *To find the areal coordinates of the centre of a conic whose equation is given in tangentials.*

The centre being the pole of the Line at Infinity has for its tangential equation

$$l(A+H+G)+m(H+B+F)+n(G+F+C)=0,$$

whence the areal coordinates of the centre are

$$\{A+H+G, \quad H+B+F, \quad G+F+C\}.$$

31. *To find the asymptotes of a conic.*

Find the pair of tangents from the centre to the conic, i.e. solve for l, m, n between

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$$

and

$$l(A+H+G)+m(H+B+F)+n(G+F+C)=0.$$

If the roots thus obtained are real, the conic is *a hyperbola.*

If the roots thus obtained are imaginary, the conic is *an ellipse.*

The condition for a parabola is that the Line at Infinity (1, 1, 1) touch the conic, i.e.

$$A+B+C+2F+2G+2H=0.$$

32. *To find the Line Coordinates of the Polar Line of* (x_1, y_1, z_1) *with respect to the conic*

$$x^2+y^2+z^2=0.$$

The equation to the polar line of the point (x_1, y_1, z_1) with respect to the above conic is

$$x_1x+y_1y+z_1z=0 \quad \text{.................(1).}$$

If the line coordinates of the line (1) be (l_1, m_1, n_1), we have

$$\frac{x_1}{l_1}=\frac{y_1}{m_1}=\frac{z_1}{n_1} \quad \text{.....................(2).}$$

33. *Dualistic Interpretation of Equations in Homogeneous Coordinates.*

Last article shews us that if we establish a geometrical proposition by means of a chain of algebraic equations in homogeneous coordinates, we have merely to replace x, y, z by l, m, n respectively throughout the chain of homogeneous equations and we shall get a new geometrical proposition, if we interpret accordingly.

This is merely the analytical form of the Geometrical Theory of Reciprocation with respect to the conic

$$x^2 + y^2 + z^2 = 0.$$

The Dual or Two-fold interpretation of equations thus explained is called the Principle of Duality.

34. EXAMPLE.

A conic circumscribes a triangle. To prove that the lines joining the vertices to the poles of the opposite sides respectively, are concurrent (Fig. 28).

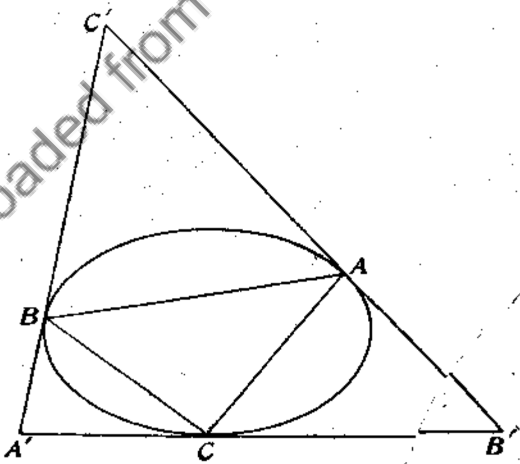

Fig. 28.

Then

$$fyz + gzx + hxy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is the equation to a circum-conic.

$$hx + fz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is the equation to the tangent at B.

$$gx + fy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is the equation to the tangent at C.

$$(-f, g, h) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

are the coordinates of A'.

$$\frac{y}{g} = \frac{z}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

is the equation to AA'.

Similarly

$$\frac{z}{h} = \frac{x}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

is the equation to BB',

and

$$\frac{x}{f} = \frac{y}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

is the equation to CC'.

$$(f, g, h) \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is the point of intersection of AA', BB', CC'.

Dualistic Proposition and Proof.

(In what follows aa' will mean the point of intersection of the lines a and a', Fig. 29.)

Replace x, y, z by l, m, n respectively in the above proof and interpret step by step.

$$fmn + gnl + hlm = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is the equation to an inscribed conic.

$$hl + fn = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is the equation to the point of contact of b.

$$gl + fm = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is the equation to the point of contact of c.

$$(-f, g, h) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

are the line coordinates of a'.

$$\frac{m}{g} = \frac{n}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

is the equation to aa'.

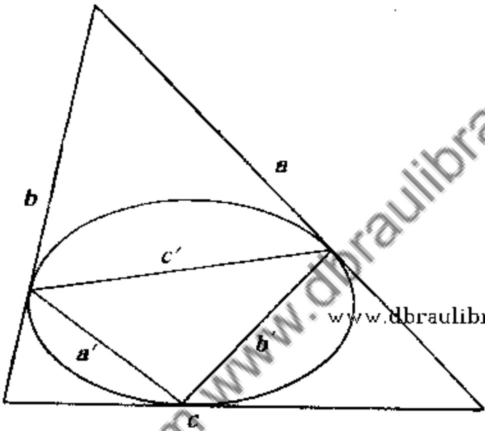

Fig. 29.

Similarly

$$\frac{n}{h} = \frac{l}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

is the equation to bb',

and

$$\frac{l}{f} = \frac{m}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

is the equation to cc'.

$$(f, g, h) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

is the line of collineation of aa', bb', cc'.

Hence we get the Dual Proposition.

Dual Proposition.

If a conic be inscribed in the triangle abc and if a', b', c' be the polars of bc, ca, ab respectively, then will aa', bb', cc' be collinear.

EXAMPLES. III.

1. Prove that the three points whose tangential equations are

$$p_1l + q_1m + r_1n = 0,$$

$$p_2l + q_2m + r_2n = 0,$$

$$p_3l + q_3m + r_3n = 0,$$

are collinear if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

2. P is a variable point on the line

$$ux + vy + wz = 0.$$

AP meets CB in Q and BP meets CA in R. Prove that QR envelopes the conic

$$umn + vnl - wlm = 0.$$

3. The conic

$$Cn^2 + 2Hlm = 0$$

touches CA at A and CB at B. A variable tangent to this conic meets CB in P and CA in Q. AP and BQ meet in R. Prove that the locus of R is the conic

$$Cxy + 2Hz^2 = 0.$$

4. A series of variable parabolas

$$Al^2 + Bm^2 + Cn^2 = 0$$

are drawn. Prove that the polars of the point (X, Y, Z) envelope the conic

$$Xmn + Ynl + Zlm = 0.$$

5. Prove that the locus of the poles with respect to the series of variable parabolas

$$Al^2 + Bm^2 + Cn^2 = 0$$

of the fixed line (l_0, m_0, n_0) is the line $(l_0^{-1}, m_0^{-1}, n_0^{-1})$.

6. A series of variable conics is drawn self-conjugate to the triangle of reference and touching the line (l_0, m_0, n_0). Prove that the locus of their centres is the line

$$xl_0^2 + ym_0^2 + zn_0^2 = 0.$$

7. The variable conic

$$Fmn + Gnl + Hlm = 0$$

touches the three sides of the triangle of reference at D, E, F respectively, and also touches the line

$$ux + vy + wz = 0.$$

Prove that the locus of the point of concurrency of the three lines AD, BE, CF is the conic

$$\frac{yz}{u} + \frac{zx}{v} + \frac{xy}{w} = 0.$$

8. If $Al^2 + Bm^2 + Cn^2 = 0$ be a parabola, prove that the axis is parallel to the line

$$x(B - C) + y(C - A) + z(A - B) = 0.$$

9. A variable conic touches the sides of the triangle of reference and also the line

$$ux + vy + wz = 0.$$

Prove that the locus of the centre is the line

$$x\left(-\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) + y\left(\frac{1}{u} - \frac{1}{v} + \frac{1}{w}\right) + z\left(\frac{1}{u} + \frac{1}{v} - \frac{1}{w}\right) = 0.$$

10. The locus of the centres of all conics touching four fixed straight lines is a straight line.

11. All parabolas inscribed in the triangle joining the middle points of the sides of a triangle are self-conjugate with respect to the original triangle.

12. P is a point on CA and Q is a point on CB such that PQ is bisected by AB. Shew that PQ envelopes the parabola whose equation is

$$2lm = n(l+m).$$

13. By referring two conics S and S' to their common self-conjugate triangle, shew that there are four conics V such that the polar reciprocal of S with respect to V is S' and *vice versa*. (St John's.)

14. Given two conics S_1 and S_2, there are four conics with respect to which S_1 and S_2 are reciprocal, and these four are such that the reciprocal of any one of the four with respect to any other of the four is a third member of the quartette. (Peterhouse etc.)

15. A and B are two fixed points, S is a fixed conic, and O is a fixed point. OPQ is a chord of S and a conic passes through $ABPQ$ such that PQ and AB are conjugate lines. Shew that if the conic meets S again in U and V, UV passes through a fixed point. (Clare etc.)

16. Prove that the general tangential equation to any point lying on the line joining the points whose tangential equations are

$$x_1l + y_1m + z_1n = 0,$$

$$x_2l + y_2m + z_2n = 0$$

is $$(x_1l + y_1m + z_1n) + \lambda(x_2l + y_2m + z_2n) = 0.$$

What equation in point coordinates is this dualistic to? (Chap. I, Art. 8.)

17. Two fixed conics touch at A and B. LM is a chord of one touching the other in Q. Prove that the line joining Q to the intersection of A and BL passes through a fixed point. (Jesus etc.)

18. If the coefficients in the equation of a right line contain a parameter λ in the first degree, the line passes through a fixed point; if in the second degree, it touches a conic.

19. The tangential equation of a conic is

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

Find the tangential coordinates of the line through the poles of two given parallel lines. Hence find the tangential equation of the centre.

20. If S, H be the foci and P, Q points on an ellipse such that SP, HQ are parallel, prove that PQ always touches an ellipse having the same principal axes.

21. A chord PQ of a conic passes through a fixed point. If the circle on PQ as diameter meets the conic again in $P'Q'$, shew that $P'Q'$ also passes through a fixed point. (Caius etc.)

22. The equation of a conic in areal coordinates is

$$x^2 - yz = 0,$$

and through the point of intersection of $x = 0$, $y = 0$ lines

$$y = mx,$$

$$(Am + B)\,y = (Cm + D)\,x$$

are drawn to intersect the conic again in P and Q respectively. If A, B, C, D be constant but m vary, shew that the envelope of PQ is a conic which touches the given conic in two points.

23. Two conics are given by the equations

$$(1)\quad \frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=0,$$

$$(2)\quad \frac{x^2}{A'}+\frac{y^2}{B'}+\frac{z^2}{C'}=0.$$

Prove that the four common tangents to (1) and (2) are

$$x\sqrt{BC'-B'C}\pm y\sqrt{CA'-C'A}\pm z\sqrt{AB'-A'B}=0,$$

and that all conics which touch these four tangents are included in

$$\frac{x^2}{A+KA'}+\frac{y^2}{B+KB'}+\frac{z^2}{C+KC'}=0.$$

(Peterhouse etc.)

24. The four common tangents to two conics meet two by two in points whose polar lines pass through one or other of the three vertices of the self-conjugate triangle of the inscribed quadrangle. (Clare etc.)

25. If two conics have three-point contact at O and Q is the pole with respect to the second of the tangent at P to the first, the envelope of PQ is a third conic.

26. Prove that four conics can be drawn to circumscribe a given triangle and to have double contact with a given conic; and that any conic to which the given triangle is self-conjugate and which touches one of the four chords of double contact touches the other three.

(Trinity.)

27. Any three points A', B', C' are taken on the sides BC, CA, AB of a triangle ABC such that

$$BA' : A'C = CB' : B'A = AC' : C'B.$$

Shew that the centroid of the triangle $A'B'C'$ coincides with that of ABC. Shew also that each side of the triangle $A'B'C'$ touches one of three fixed parabolas respectively.

28. The conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

meets the sides in three point pairs, and these on being joined to the respectively opposite angular points of the triangle of reference, determine three line pairs; prove that the six lines thus constructed touch the conic

$$bcl^2 + cam^2 + abn^2 - 2afmn - 2bgnl - 2chlm = 0.$$

(St John's.)

29. Find the tangential equation of all conics touching the four lines

$$lx \pm my \pm nz = 0,$$

and shew that if

$$L\lambda^2 + M\mu^2 + N\nu^2 = 0$$

be a conic touching all four lines, then the cross-ratio of the range determined by the four given tangents on any fifth is the ratio with the sign changed of some two of the three quantities Ll^2, Mm^2, Nn^2. (St John's.)

30. A conic is defined by the tangential equation

$$\phi(l, m, n) = 0.$$

Prove that the tangential coordinates of its asymptotes are given by

$$\phi(l, m, n) = 0,$$

$$\frac{\partial\phi}{\partial l} + \frac{\partial\phi}{\partial m} + \frac{\partial\phi}{\partial n} = 0.$$

(Peterhouse etc.)

31. A conic circumscribes a triangle and its centre moves along a median. Prove that the asymptotes touch a conic which touches two of the sides of the triangle at the extremities of the remaining side. (Math. Trip.)

32. The line of collinearity of the middle points of the diagonals of a quadrilateral is drawn, and the middle point of the intercept on it between any two sides is joined to the point in which they intersect. Shew that the six lines so constructed together with the line of collinearity and the three diagonals themselves touch a parabola. (Math. Trip.)

33. Through each point of the line joining the middle points of the diagonals of a quadrilateral, the diameter conjugate to the line of middle points with respect to the conic touching the four sides of the quadrilateral and having that point as centre is drawn. Shew that these diameters envelope a parabola. (Clare etc.)

34. A conic passes through three given points and one of its asymptotes is in a fixed direction; prove that the other asymptote touches a fixed parabola, which touches the lines joining the three points, and has its axis in the fixed direction. Find also the locus of the centre of the conic. (Clare etc.)

35. From $(x_0 y_0)$ four normals are drawn to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

Shew that the four corresponding tangents satisfy the tangential equation

$$x_0 mn - y_0 nl + c^2 lm = 0. \quad (c^2 = a^2 - b^2.)$$

(St John's.)

36. Normals from $(x_0 y_0)$ are drawn to

$$y^2 - 4ax = 0,$$

and $$lx + my + n = 0$$

is one of the tangents at the feet of the normals. Prove that

$$l^2y_0 - lm(x_0 - 2a) + mn =$$

and therefore all three tangents touch the parabola

$$(x + x_0 - 2a)^2 + 4yy_0 = 0.$$ (St John's.)

37. A parabola touches the sides of the triangle ABC in D, E, F respectively, and DE, DF cut the diameter through A in b, c respectively. Shew that Bb and Cc are parallel. (Sidney Sussex.)

38. A conic passes through three fixed points and touches a fixed straight line. Shew that the locus of the pole of the line joining two of the points is another conic touching each of the lines joining the points.

39. A series of parabolas touch the three sides of a triangle. Prove that the line joining the points of contact of any two sides passes through a fixed point. (Peterhouse.)

40. Shew that the condition of contact of the line

$$lx + my + nz = 0$$

with the curve

$$y^2z^2 + z^2x^2 + x^2y^2 - 2x^2yz - 2y^2zx - 2z^2xy = 0$$

is $$(l + m + n)^3 = 27lmn.$$ (Jesus etc.)

41. If the point coordinates be areal, then the asymptotes of all conics circumscribing the fundamental triangle and passing through the point $(x_1y_1z_1)$ touch a curve of the third class whose equation is

$$\frac{(m-n)^2 l}{x_1} + \frac{(n-l)^2 m}{y_1} + \frac{(l-m)^2 n}{z_1} = 0.$$

(Math. Trip.)

CHAPTER IV

THE CIRCULAR POINTS AT INFINITY

1. The following properties in Projective Geometry will now be required:

Circle. **All circles cut the Line at Infinity in the same two points, usually denoted by I and J, and called "The Circular Points at Infinity"** [25].

Concentric Circles. **A series of Concentric Circles having O as their common centre are to be regarded as a series of conics touching OI and OJ at I and J respectively** [26].

Perpendicular Lines. **If OP and OQ be two perpendicular lines, they will harmonically separate OI and OJ** [27].

Rectangular Hyperbola. **A Rectangular Hyperbola cuts the Line at Infinity in two points that harmonically separate I and J.**

Focus. **If S be a focus of a conic, then SI and SJ are each tangents to the conic** [28].

2. *To find the areal equation to the Circum-Circle of the triangle of reference.*

Let a_0, b_0, c_0, A_0, B_0, C_0 denote the magnitudes of the sides and angles of the triangle of reference (Fig. 30) as in Chap. I, Art. 5.

The equation to any circum-conic is of the form

$$fyz + gzx + hxy = 0 \quad \text{..............(1)},$$

and the tangent at C to this conic will be

$$gx + fy = 0 \quad \text{....................(2)}.$$

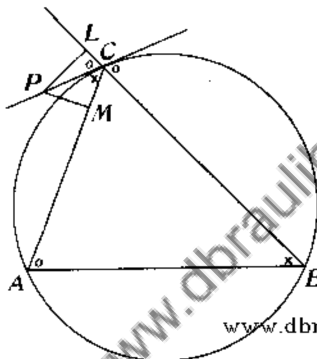

Fig. 30.

Let P be a point on the tangent at C to the Circum-Circle.

Drop the perpendiculars PL and PM on CB and CA respectively.

Then if $P \equiv (x, y, z)$ in areal coordinates, and if we observe that P lies on the positive side of CB and on the negative side of CA in the figure, we get

$$\frac{x}{-y} = \frac{a_0 \times PL}{b_0 \times PM} = \frac{a_0 \times PC \sin A_0}{b_0 \times PC \sin B_0} = \frac{a_0^2}{b_0^2}.$$

$$\therefore \qquad b_0^2 x + a_0^2 y = 0 \quad \text{..................(3)}.$$

Comparing (2) and (3) we get

$$\frac{f}{a_0^2} = \frac{g}{b_0^2}$$

$$= \frac{h}{c_0^2} \qquad \text{(by symmetry)}.$$

Hence the equation to the Circum-Circle is

$$a_0^2yz + b_0^2zx + c_0^2xy = 0.$$

3. *To find the combined tangential equation to I and J, the Circular Points at Infinity.*

We here use the theorem that "all circles cut the Line at Infinity in the same two points, I and J."

Let (x_1, y_1, z_1) be the areal coordinates of I (or J).

Let (l, m, n) be a line through that point.

Since I lies on the line (l, m, n) and also on the Line at Infinity,

$$\therefore \quad lx_1 + my_1 + nz_1 = 0 \quad \text{..............(1)},$$

$$x_1 + y_1 + z_1 = 0 \quad \text{..............(2)}.$$

Solving we get

$$\frac{x_1}{m-n} = \frac{y_1}{n-l} = \frac{z_1}{l-m} \quad \text{..............(3)}.$$

Since this point lies on the Circum-Circle we get, on substituting from (3) in

$$a_0^2yz + b_0^2zx + c_0^2xy = 0,$$

$$a_0^2(n-l)(l-m) + b_0^2(l-m)(m-n) + c_0^2(m-n)(n-l) = 0,$$

i.e. on expansion

$$a_0^2l^2 + b_0^2m^2 + c_0^2n^2$$
$$-2b_0c_0 \cos A_0mn - 2c_0a_0 \cos B_0nl - 2a_0b_0 \cos C_0lm = 0,$$

which is the combined tangential equation to I and J.

4. *To find the condition that the lines (l_1, m_1, n_1) and (l_2, m_2, n_2) be perpendicular.*

Let the given lines be OP_1 and OP_2.

Since OP_1 and OP_2 are perpendicular, they will harmonically separate OI and OJ. Now OI and OJ may be regarded as the tangents from O to the degenerate point-conic I; J.

Hence OP_1 and OP_2 are conjugate lines with respect to the point-conic I; J whose tangential equation is

$$a_0^2l^2 + b_0^2m^2 + c_0^2n^2$$
$$- 2b_0c_0 \cos A_0 mn - 2c_0a_0 \cos B_0 nl - 2a_0b_0 \cos C_0 lm = 0,$$

the condition for which is (Chap. III, Art. 17)

$$a_0^2l_1l_2 + b_0^2m_1m_2 + c_0^2n_1n_2 - b_0c_0 \cos A_0 (m_1n_2 + m_2n_1)$$
$$- c_0a_0 \cos B_0 (n_1l_2 + n_2l_1) - a_0b_0 \cos C_0 (l_1m_2 + l_2m_1) = 0.$$

5. *To find the condition that the conic represented by the general equation of the second degree be a circle.*

Since a circle passes through I and J, it must pass through the points in which the Line at Infinity cuts the Circum-Circle. Hence the general equation to a circle will be (Chap. II, Art. 11, Cor.)

$$a_0^2yz + b_0^2zx + c_0^2xy + (x + y + z)(lx + my + nz) = 0 \ldots (1).$$

Identifying this equation with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots (2)$$

we get that the corresponding coefficients will be proportional, and hence

$$l = \lambda a,$$
$$m = \lambda b,$$
$$n = \lambda c,$$
$$m + n + a_0^2 = 2\lambda f,$$
$$n + l + b_0^2 = 2\lambda g,$$
$$l + m + c_0^2 = 2\lambda h,$$

whence it is plain that

$$\frac{b + c - 2f}{a_0^2} = \frac{c + a - 2g}{b_0^2} = \frac{a + b - 2h}{c_0^2}$$

which are the *two* conditions required.

6. *To find the condition that the general equation of the second degree may represent a Rectangular Hyperbola.*

By Chap. IV, Art. 1, the Circular Points at Infinity I and J will be conjugate points with respect to the Rectangular Hyperbola.

Let $I \equiv (x_1, y_1, z_1)$ and $J \equiv (x_2, y_2, z_2)$.

Then the combined tangential equation to I and J will be

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2) = 0 \;\ldots\ldots(1).$$

But the combined tangential equation to I and J is known from Chap. IV, Art. 3, to be

$$a_0^2l^2 + b_0^2m^2 + c_0^2n^2$$
$$-2b_0c_0 \cos A_0mn - 2c_0a_0 \cos B_0nl - 2a_0b_0 \cos C_0lm = 0 \ldots(2).$$

Hence the corresponding coefficients in (1) and (2) must be proportional. Thus

$$\begin{aligned} x_1x_2 &= \rho a_0^2 \\ y_1y_2 &= \rho b_0^2 \\ z_1z_2 &= \rho c_0^2 \\ y_1z_2 + y_2z_1 &= -2\rho b_0c_0 \cos A_0 \quad \ldots\ldots\ldots(3), \\ z_1x_2 + z_2x_1 &= -2\rho c_0a_0 \cos B_0 \\ x_1y_2 + x_2y_1 &= -2\rho a_0b_0 \cos C_0 \end{aligned}$$

where ρ is a numerical multiplier.

Now $I \equiv (x_1, y_1, z_1)$ and $J \equiv (x_2, y_2, z_2)$ will be conjugate points with respect to the given Rectangular Hyperbola

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

if (Chap. II, Art. 3, Case II)

$$ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1)$$
$$+ g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1) = 0,$$

i.e. if

$$aa_0^2 + bb_0^2 + cc_0^2 - 2fb_0c_0 \cos A_0 - 2gc_0a_0 \cos B_0 \\ - 2ha_0b_0 \cos C_0 = 0,$$

which is therefore the condition required.

COROLLARY.

If the combined point equation to a pair of straight lines be given and if we wish to find the condition that they be perpendicular, we regard them as a rectangular hyperbola and use the foregoing condition.

7. *Radical Axis of two given Circles.*

If S be the equation to a circle, then

$$\lambda S + (lx + my + nz)(x + y + z) = 0 \quad \ldots\ldots\ldots(1)$$

will represent the equation to another circle S′, and

$$lx + my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

will be the equation to their Radical Axis.

The equation (1) represents a circle, since it passes through the intersections of the Line at Infinity

$$x + y + z = 0$$

and the circle $\qquad S = 0,$

i.e. it passes through I and J.

Also the common chord of the circles S and S' other than the Line at Infinity is

$$lx + my + nz = 0,$$

which is therefore their Radical Axis.

If we take S to be the Circum-Circle we can find very convenient interpretations for l, m, n, as follows:

8. *If a circle be taken in the form*

$$a_0^2yz + b_0^2zx + c_0^2xy = (x+y+z)(lx+my+nz) \quad \ldots(1),$$

to find a geometrical interpretation for l, m, n.

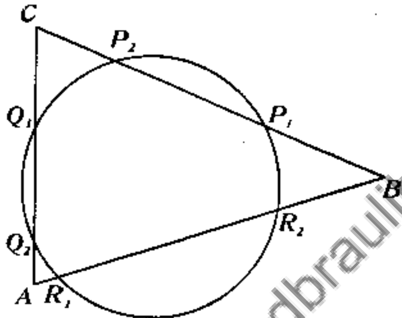

Fig. 31.

Let the above circle cut the sides of the triangle of reference in the points P_1, P_2; Q_1, Q_2; R_1, R_2 respectively (Fig. 31).

Let $(x, y, 0)$ be the "actual areal coordinates" of one of the points in which the given circle cuts AB.

Then $$x + y = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $$lx^2 + (l + m - c_0^2)\,xy + my^2 = 0 \quad \ldots\ldots\ldots(3)$$

if we substitute in (1).

Hence eliminating y between (1) and (2)

$$lx^2 + (l + m - c_0^2)\,x(1-x) + m(1-x)^2 = 0 \quad \ldots(4).$$

Thus if x_1 and x_2 be the roots of the equation (4)

$$x_1x_2 = \frac{m}{c_0^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But $$x_1 = \frac{\Delta CR_1B}{\Delta CAB} = \frac{R_1B}{c_0}, \text{ etc.}$$

$$\therefore \quad R_1B \,.\, R_2B = m \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Now if we denote by t_1, t_2, t_3 the lengths of the tangents from A, B, C respectively to the circle, we get from (6) $t_2^2 = R_1B \, . \, R_2B = m$.

Thus $$l = t_1^2, \quad m = t_2^2, \quad n = t_3^2,$$

and we may write the equation to the given circle thus:

$$a_0^2yz + b_0^2zx + c_0^2xy = (x+y+z)(t_1^2x + t_2^2y + t_3^2z),$$

a formula which enables us to write down the equations to most circles almost instantaneously.

9. By the formula of Art. 8 we get the equations to the following circles very quickly and we shall place them here for convenience of reference.

Nine-Points Circle.

$$2(a_0^2yz + b_0^2zx + c_0^2xy)$$
$$= (x+y+z)(b_0c_0 \cos A_0 x + c_0a_0 \cos B_0 y + a_0b_0 \cos C_0 z).$$

Polar Circle (i.e. circle to which ABC is self-conjugate).

$$a_0^2yz + b_0^2zx + c_0^2xy$$
$$= (x+y+z)(b_0c_0 \cos A_0 x + c_0a_0 \cos B_0 y + a_0b_0 \cos C_0 z)$$

or $$\cot A_0 x^2 + \cot B_0 y^2 + \cot C_0 z^2 = 0.$$

Inscribed Circle.

$$(s_0 - a_0)^2 x^2 + (s_0 - b_0)^2 y^2 + (s_0 - c_0)^2 z^2 - 2(s_0 - b_0)(s_0 - c_0) yz$$
$$- 2(s_0 - c_0)(s_0 - a_0) zx - 2(s_0 - a_0)(s_0 - b_0) xy = 0.$$

Escribed Circles.

Opposite the vertex A.

$$s_0^2x^2 + (s_0 - c_0)^2 y^2 + (s_0 - b_0)^2 z^2 - 2(s_0 - b_0)(s_0 - c_0) yz$$
$$+ 2s_0(s_0 - b_0) zx + 2s_0(s_0 - c_0) xy = 0 \text{; etc.}$$

10. Example 1.

To prove that all rectangular hyperbolas circumscribing the triangle of reference pass through the orthocentre.

The conic $$fyz + gzx + hxy = 0 \quad \text{..............(1)}$$

will be a rectangular hyperbola if (Chap. IV, Art. 6)

$$fb_0c_0 \cos A_0 + gc_0a_0 \cos B_0 + ha_0b_0 \cos C_0 = 0,$$

i.e. if $$f \cot A_0 + g \cot B_0 + h \cot C_0 = 0 \quad \text{.........(2)},$$

which is the condition that the orthocentre

$$(\tan A_0, \ \tan B_0, \ \tan C_0)$$

lies on the conic (1).

Example 2.

The locus of the centres of all rectangular hyperbolas passing through the vertices of the triangle of reference is the Nine-Points Circle.

$$fyz + gzx + hxy = 0$$

will be a rectangular hyperbola if

$$f \cot A_0 + g \cot B_0 + h \cot C_0 = 0 \quad \text{......(1)}.$$

The centre of the curve is given by

$$hy + gz = fz + hx = gx + fy \quad \text{...........(2)}.$$

Eliminating f, g, h between (1) and (2), we get

$$2(\cot A_0 x^2 + \cot B_0 y^2 + \cot C_0 z^2)$$
$$= (x + y + z)(\cot A_0 x + \cot B_0 y + \cot C_0 z),$$

which is another form for the equation to the Nine-Points Circle and can easily be verified by shewing that it passes through the feet of the perpendiculars and the mid-points of the sides and is thus uniquely defined.

11. *To find the tangential equation to a circle having $O \equiv (x_1, y_1, z_1)$ as centre.*

We use the property of Projective Geometry that a circle having O for centre touches OI and OJ at I and J respectively.

Hence we must find the general tangential equation to all conics having double contact with the Point-Conic I; J at the points where it is met by the tangents to it from O, i.e. the required circle will be of the form (Chap. III, Art. 28)

$$(lx_1 + my_1 + nz_1)^2 + \lambda (a_0^2 l^2 + b_0^2 m^2 + c_0^2 n^2 - 2b_0 c_0 \cos A_0 mn - 2c_0 a_0 \cos B_0 nl - 2a_0 b_0 \cos C_0 lm) = 0.$$

For various values of λ the above equation may be considered as the general tangential equation to a series of concentric circles having $O \equiv (x_1, y_1, z_1)$ for centre.

12. *To find the tangential equation to a circle of radius ρ having $O \equiv (x_1, y_1, z_1)$ as centre.*

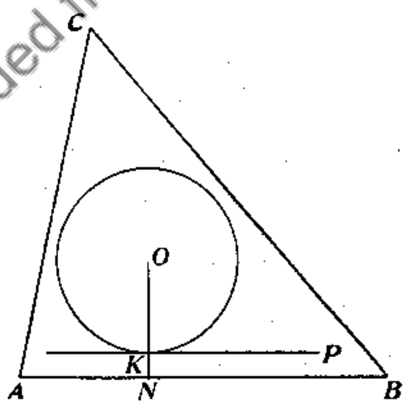

Fig. 32.

By last article its tangential equation will be of the form

$$(lx_1+my_1+nz_1)^2=\lambda(a_0^2l^2+b_0^2m^2+c_0^2n^2-2b_0c_0\cos A_0mn$$
$$-2c_0a_0\cos B_0nl-2a_0b_0\cos C_0lm)\ldots(1).$$

Let (x_1, y_1, z_1) be actual areal coordinates.

From O draw ON perpendicular to AB meeting the circle in K (Fig. 32).

Let the tangent at K be KP and let $K\equiv(x_2, y_2, z_2)$ in "actual areal coordinates."

Then the equation to KP will be $z=z_2$,

i.e. $$1=zz_2^{-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

i.e. $$x+y+z(1-z_2^{-1})=0\ldots\ldots\ldots\ldots\ldots(3),$$

if we render the equation (2) homogeneous by means of

$$x+y+z=1.$$

The line (2) will touch (1) if

$$(1-z_1z_2^{-1})^2=\lambda\{a_0^2+b_0^2+c_0^2(1-z_2^{-1})^2$$
$$-2c_0(b_0\cos A_0+a_0\cos B_0)(1-z_2^{-1})-2a_0b_0\cos C_0\},$$

i.e. if

$$(1-z_1z_2^{-1})^2=\lambda c_0^2\{1+(1-z_2^{-1})^2-2(1-z_2^{-1})\}$$

(since $c_0^2=a_0^2+b_0^2-2a_0b_0\cos C_0$ and $c_0=a_0\cos B_0+b_0\cos A_0$),

i.e. if $$\lambda=\left(\frac{z_2}{c_0}-\frac{z_1}{c_0}\right)^2,$$

i.e. if $$\lambda=\frac{(ON-KN)^2}{4\Delta^2},\ \because\ z_1=\frac{c_0ON}{2\Delta}\text{, etc.,}$$

i.e. if $$\lambda=\frac{\rho^2}{4\Delta^2}.$$

Hence the required equation to the given circle is

$$4\Delta^2 (lx_1 + my_1 + nz_1)^2 = \rho^2 (a_0^2 l^2 + b_0^2 m^2 + c_0^2 n^2 - 2b_0 c_0 \cos A_0 mn - 2c_0 a_0 \cos B_0 nl - 2a_0 b_0 \cos C_0 lm).$$

If we wish to remove the condition that x_1, y_1, z_1 be actual areal coordinates, we can write the equation thus:

$$4\Delta^2 (lx_1 + my_1 + nz_1)^2 = \rho^2 (x_1 + y_1 + z_1)^2 (a_0^2 l^2 + b_0^2 m^2 + c_0^2 n^2 - 2b_0 c_0 \cos A_0 mn - 2c_0 a_0 \cos B_0 nl - 2a_0 b_0 \cos C_0 lm).$$

13. The tangential equations to the most oft-recurring circles can be put down in the above form by first finding their centre, writing down the equation of Chap. IV, Art. 11, and finding λ by expressing the condition that the circle required touch some given line or satisfy some other suitable condition. The following are the most common:

Inscribed Circle.

$$\cot \frac{A_0}{2} mn + \cot \frac{B_0}{2} nl + \cot \frac{C_0}{2} lm = 0.$$

Escribed Circle (opposite the vertex A).

$$\cot \frac{A_0}{2} mn - \tan \frac{B_0}{2} nl - \tan \frac{C_0}{2} lm = 0.$$

Circum-Circle.

$$R^2 (l \sin 2A_0 + m \sin 2B_0 + n \sin 2C_0)^2 = (a_0^2 l^2 + b_0^2 m^2 + c_0^2 n^2 - 2b_0 c_0 \cos A_0 mn - 2c_0 a_0 \cos B_0 nl - 2a_0 b_0 \cos C_0 lm).$$

Polar Circle (i.e. circle to which ABC is self-conjugate).

$$\tan A_0 l^2 + \tan B_0 m^2 + \tan C_0 n^2 = 0.$$

14. *To find the equation of the Radical Axis to two Circles.*

Taking the two circles in the form

$$a_0^2 yz + b_0^2 zx + c_0^2 xy = (x + y + z)(t_1^2 x + t_2^2 y + t_3^2 z),$$

$$(a_0^2 yz + b_0^2 zx + c_0^2 xy) = (x + y + z)(t_1'^2 x + t_2'^2 y + t_3'^2 z)$$

(Chap. IV, Art. 8), we see on subtracting that they cut in the points where they are met by the lines

$$(x+y+z)\left(\overline{t_1^2 - t_1'^2}\,x + \overline{t_2^2 - t_2'^2}\,y + \overline{t_3^2 - t_3'^2}\,z\right) = 0,$$

one of which is the equation to the Line at Infinity and consequently the other will be the Radical Axis.

15. *Foci.*

To find the general tangential equation to conics confocal with a given conic Σ.

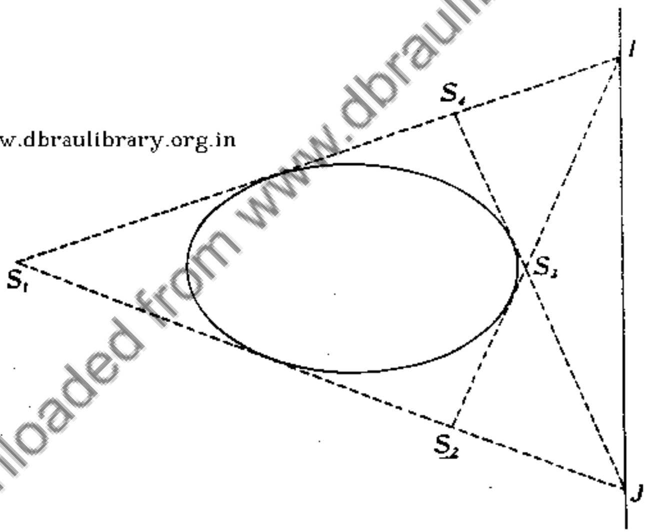

Fig. 33.

We use the property stated in Chap. IV. Art. 1, that if S be a focus of a conic, SI and SJ will each touch the conic. Hence if we draw the four tangents from I and J to the conic Σ (Fig. 33) we shall get the four foci, and all conics confocal with Σ will therefore touch the four common tangents to Σ and the degenerate point-conic I; J. Hence the tangential equation to any conic confocal with Σ must be of the form

$$\lambda\Sigma + (a_0^2l^2 + b_0^2m^2 + c_0^2n^2 - 2b_0c_0 \cos A_0mn$$
$$- 2c_0a_0 \cos B_0nl - 2a_0b_0 \cos C_0lm) = 0,$$

by Chap. III, Art. 28, Cor. I.

16. *To find the general tangential equation to conics having two given points* $S_1 \equiv (x_1, y_1, z_1)$ *and* $S_2 \equiv (x_2, y_2, z_2)$ *as foci.*

We have to write down the general tangential equation to all conics touching the four common tangents of the two point-conics S_1; S_2 and I; J, namely,

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2) + \lambda(a_0^2l^2 + b_0^2m^2 + c_0^2n^2$$
$$- 2b_0c_0 \cos A_0mn - 2c_0a_0 \cos B_0nl - 2a_0b_0 \cos C_0lm) = 0.$$

17. *To find the general tangential equation to all parabolas having a given focus* S_1.

In this case S_2 of last article is at infinity and we therefore have the equation of last article coupled with the condition

$$x_2 + y_2 + z_2 = 0.$$

COROLLARY.

(x_2, y_2, z_2) *will be the point of contact of the parabola and the Line at Infinity.*

18. EXAMPLE 1.

To prove that the Nine-Points Circle touches the Inscribed Circle and the three Escribed Circles.

Consider the Nine-Points Circle and the In-Circle.

We take in accordance with Chap. IV, Art. 14, the Radical Axis of the two circles in the form

$$(t_1^2 - t_1'^2)\,x + (t_2^2 - t_2'^2)\,y + (t_3^2 - t_3'^2)\,z = 0,$$

which for the above two circles is easily found to be

$$\frac{x}{b_0 - c_0} + \frac{y}{c_0 - a_0} + \frac{z}{a_0 - b_0} = 0 \;\ldots\ldots\ldots\ldots(1).$$

The line (1) touches the In-Circle

$$\cot\frac{A_0}{2}\,mn + \cot\frac{B_0}{2}\,nl + \cot\frac{C_0}{2}\,lm = 0 \;\ldots\ldots(2),$$

since

$$(b_0 - c_0)\cot\frac{A_0}{2} + (c_0 - a_0)\cot\frac{B_0}{2} + (a_0 - b_0)\cot\frac{C_0}{2} = 0 \ldots(3).$$

Hence since the Radical Axis touches the one circle it touches the other also.

EXAMPLE 2.

To show that the locus of the foci of all parabolas touching the three sides of a triangle is the Circum-Circle of the triangle.

Taking the given triangle as triangle of reference, the parabola having (x_1, y_1, z_1) for focus and touching the line at infinity at (x_2, y_2, z_2) will by Chap. IV, Art. 17, be

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2)$$
$$+ \lambda\,(a_0^2 l^2 + b_0^2 m^2 + c_0^2 n^2 - 2b_0c_0\cos A_0\,mn - 2c_0a_0\cos B_0\,nl$$
$$- 2a_0b_0\cos C_0\,lm) = 0 \ldots(1),$$

where $$x_2 + y_2 + z_2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

(1) will touch the sides of the triangle of reference if

$$x_1x_2 + \lambda a_0^2 = 0, \quad y_1y_2 + \lambda b_0^2 = 0, \quad z_1z_2 + \lambda c_0^2 = 0 \ldots (3).$$

Substitute from (3) in (2) for x_2, y_2, z_2 and we get as the locus of the point (x_1, y_1, z_1)

$$\frac{a_0^2}{x_1} + \frac{b_0^2}{y_1} + \frac{c_0^2}{z_1} = 0,$$

i.e. the Circum-Circle.

EXAMPLE 3.

To find the equation to the directrix of the parabola

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0 \ldots (1).$$

The directrix of a parabola may be looked upon as the locus of points the tangents from which to the parabola are perpendicular.

Thus if O be a point and OP and OQ be the tangents therefrom to the parabola, and if OP and OQ harmonically separate OI and OJ, O will be a point on the directrix.

Now if O be a point on the Line at Infinity, it is plain that the tangents from O to the parabola must be considered as harmonically separating OI and OJ, since one of the tangents coincides with OI and OJ. Hence the Line at Infinity will appear as part of the locus. The tangents from the point (x_1, y_1, z_1) to the conic (1) will be (Chap. II, Art. 3, Case III)

$$(a^2x_1^2 + b^2y_1^2 + c^2z_1^2 - 2bcy_1z_1 - 2caz_1x_1 - 2abx_1y_1)$$
$$(a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy)$$
$$= \{ax(ax_1 - by_1 - cz_1) + by(-ax_1 + by_1 - cz_1)$$
$$+ cz(-ax_1 - by_1 + cz_1)\}^2 \ldots (2).$$

These two tangents will be perpendicular if (Chap. IV, Art. 6)

$$(a^2x_1^2 + b^2y_1^2 + c^2z_1^2 - 2bcy_1z_1 - 2caz_1x_1 - 2abx_1y_1)(a^2a_0^2 + b^2b_0^2$$
$$+ c^2c_0^2 + 2bcb_0c_0 \cos A_0 + 2cac_0a_0 \cos B_0 + 2aba_0b_0 \cos C_0)$$
$$= a_0^2a^2(ax_1 - by_1 - cz_1)^2 + b_0^2b^2(-ax_1 + by_1 - cz_1)^2$$
$$+ c_0^2c^2(-ax_1 - by_1 + cz_1)^2$$
$$- 2bcb_0c_0 \cos A_0(-ax_1 + by_1 - cz_1)(-ax_1 - by_1 + cz_1)$$
$$- 2cac_0a_0 \cos B_0(-ax_1 - by_1 + cz_1)(ax_1 - by_1 - cz_1)$$
$$- 2aba_0b_0 \cos C_0(ax_1 - by_1 - cz_1)(-ax_1 + by_1 - cz_1) \ldots (3).$$

Now this equation must be the combined equation to the directrix

$$lx + my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and the Line at Infinity

$$x + y + z = 0,$$

and must therefore be of the form

$$(lx + my + nz)(x + y + z) = 0 \quad \ldots\ldots\ldots (5)$$

if we use the condition for (1) being a parabola.

Hence, if we change x_1, y_1, z_1 in (3) to the current coordinates x, y, z, then to find the coefficients of (4) we need only find the coefficients of x^2, y^2, z^2 respectively in (3), as we see on identifying (3) and (5). The directrix will therefore be

$$a \cot A_0 x + b \cot B_0 y + c \cot C_0 z = 0.$$

EXAMPLES. IV.

1. Prove the following areal coordinates

(1) Orthocentre $\equiv (\tan A_0, \tan B_0, \tan C_0)$.

(2) Circumcentre $\equiv (\sin 2A_0, \sin 2B_0, \sin 2C_0)$.

(3) Nine-Points Centre $\equiv$

$$(a_0 \cos \overline{B_0 - C_0}, b_0 \cos \overline{C_0 - A_0}, c_0 \cos \overline{A_0 - B_0}).$$

2. Hence shew that the Orthocentre, the Circumcentre, the Nine-Points Centre and the Centroid of a triangle are in the same straight line, the line of collinearity being

$$x \cos A_0 \sin \overline{B_0 - C_0} + y \cos B_0 \sin \overline{C_0 - A_0} + z \cos C_0 \sin \overline{A_0 - B_0} = 0.$$

3. Prove that the internal bisectors of the angles of the triangle of reference are

$$\frac{y}{b_0} = \frac{z}{c_0},$$

$$\frac{z}{c_0} = \frac{x}{a_0},$$

$$\frac{x}{a_0} = \frac{y}{b_0}.$$

Hence shew that the three internal bisectors are concurrent and find their point of intersection.

4. Prove that the three external bisectors of the angles of the triangle of reference are

$$\frac{y}{b_0} + \frac{z}{c_0} = 0,$$

$$\frac{z}{c_0} + \frac{x}{a_0} = 0,$$

$$\frac{x}{a_0} + \frac{y}{b_0} = 0.$$

5. Hence shew that the three points, where the external bisectors of the angles of a triangle respectively meet the opposite sides, are collinear and find the equation to the line of collinearity.

6. Prove that the three points, where the tangents to the circum-circle at the vertices of the triangle of reference respectively meet the opposite sides, all lie on the line

$$\frac{x}{a_0^2} + \frac{y}{b_0^2} + \frac{z}{c_0^2} = 0.$$

7. Prove that the equations to the three perpendiculars from the vertices on the opposite sides are respectively

$$x \cot A_0 = y \cot B_0,$$

$$y \cot B_0 = z \cot C_0,$$

$$z \cot C_0 = x \cot A_0.$$

8. Shew that the equations to the three perpendiculars drawn through the mid-points of the sides are respectively

$$\frac{\sin (B_0 - C_0)}{\sin A_0} x + y - z = 0,$$

$$-x + \frac{\sin (C_0 - A_0)}{\sin B_0} y + z = 0,$$

$$+x - y + \frac{\sin (A_0 - B_0)}{\sin C_0} z = 0;$$

and prove from these equations that the lines are concurrent.

9. If A', B', C' be the poles of the sides of the triangle of reference with respect to the circum-circle, the equation to the circum-circle of the triangle $A'B'C'$ will be

$$(a_0^2 yz + b_0^2 zx + c_0^2 xy) + R^2 \tan A_0 \tan B_0 \tan C_0 (x + y + z)(x \cot A_0 + y \cot B_0 + z \cot C_0) = 0.$$

10. Prove that the lines

$$x \cot A_0 + y \cot B_0 + z \cot C_0 = 0,$$

$$x \cos A_0 \sin (B_0 - C_0) + y \cos B_0 \sin (C_0 - A_0) + z \cos C_0 \sin (A_0 - B_0) = 0$$

are perpendicular.

11. Prove that the equation to the circle through A, B and the orthocentre is

$$a_0^2 yz + b_0^2 zx + c_0^2 xy = 2a_0 b_0 \cos C_0 z (x + y + z).$$

12. Prove that the equation to the circle which passes through the centres of the escribed circles of the triangle of reference is

$$b_0 c_0 x^2 + c_0 a_0 y^2 + a_0 b_0 z^2 + (a_0 + b_0 + c_0)(a_0 yz + b_0 zx + c_0 xy) = 0.$$

13. Prove that the normal at C to the circum-conic

$$fyz + gzx + hxy = 0$$

is

$$\frac{x}{a_0^2 g - a_0 b_0 \cos C_0 f} = \frac{y}{b_0^2 f - a_0 b_0 \cos C_0 g}.$$

14. Prove that the equation to the rectangular hyperbola circumscribing the triangle of reference and passing through the centroid is

$$a_0 \sin (B_0 - C_0)\, yz + b_0 \sin (C_0 - A_0)\, zx + c_0 \sin (A_0 - B_0)\, xy = 0.$$

15. Prove that the centre of the inscribed circle of a triangle lies on any rectangular hyperbola circumscribing the triangle formed by joining the centres of the escribed circles. (Pembroke etc.)

16. A triangle is inscribed in a circle, and any point P on the circumference is joined to the orthocentre of the triangle. Shew that the joining line is bisected by the pedal line of the point P.

17. I is the centre of the inscribed circle of a triangle and I_1, I_2, I_3 the centres of the escribed circles. Shew that II_1, II_2, II_3 are bisected by the circumference of the circum-circle.

18. ABC is a triangle and I_2, I_3 are the centres of the escribed circles which touch AC and AB respectively; shew that the points B, C, I_2, I_3 lie upon a circle whose centre is on the circumference of the circum-circle of the triangle ABC.

19. In a triangle ABC, I is the centre of the inscribed circle; shew that the centres of the circles circumscribed about the triangles BIC, CIA, AIB lie on the circumference of the circle circumscribed about the given triangle.

20. The centre of the circle inscribed in any triangle self-conjugate with respect to a rectangular hyperbola lies on that hyperbola.

21. The equation of the line through (x', y', z') parallel to (l, m, n) is in areals

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ m-n & n-l & l-m \end{vmatrix} = 0.$$

22. The equation of the line through (x', y', z') perpendicular to (l, m, n) is in areals

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ a_0(a_0 l - b_0 \cos C_0 m - c_0 \cos B_0 n), & b_0(b_0 m - c_0 \cos A_0 n - a_0 \cos C_0 l), & c_0(c_0 n - a_0 \cos B_0 l - b_0 \cos A_0 m) \end{vmatrix} = 0.$$

23. Shew that the locus of the centre of a conic which circumscribes the triangle of reference and touches the line

$$lx + my + nz = 0$$

is $\sqrt{lx(-x+y+z)} + \sqrt{my(x-y+z)} + \sqrt{nz(x+y-z)} = 0.$

(Peterhouse etc.)

24. Shew that the areal coordinates of the centre of the rectangular hyperbola of a system of four-point conics, referred to their common self-conjugate triangle, are given by

$$(b_0^2z'^2 - c_0^2y'^2)\,x = (c_0^2x'^2 - a_0^2z'^2)\,y = (a_0^2y'^2 - b_0^2x'^2)\,z,$$

where (x', y', z') are the coordinates of the points in areal coordinates. (Clare etc.)

25. Shew that the equations of the two parabolas circumscribing the triangle of reference, and touching the circumscribed circle at A are

$$(b_0 \pm c_0)^2\,yz + x\,(b_0^2z + c_0^2y) = 0.$$

(Clare etc.)

26. A conic has its centre upon another conic and is inscribed in a triangle which is self-conjugate with regard to the second conic. Prove that if one of the conics is a circle, the other is a rectangular hyperbola. (Jesus etc.)

27. Prove that the equation of the circle on the line joining the centre of gravity and the orthocentre of the triangle as diameter is

$$3\,(a_0\beta\gamma + b_0\gamma\alpha + c_0\alpha\beta)$$
$$= 2\,(\alpha\cos A_0 + \beta\cos B_0 + \gamma\cos C_0)\,(a_0\alpha + b_0\beta + c_0\gamma)$$

in trilinear coordinates. (Jesus etc.)

28. Shew that the lines joining the centres of the escribed circles of a triangle to the corresponding vertices of the pedal triangle are concurrent. (Pembroke etc.)

29. A circle and a rectangular hyperbola meet in the four points A, B, C, D. The centre of the circle lies on AB. Prove that the centre of the rectangular hyperbola lies on CD. (Magdalene.)

30. The sides BC, CA, AB of a triangle are cut by the internal bisectors of the angles in D, E, F and by the external bisectors in A', B', C'. Shew that $A'B'C'$, $A'EF$, $B'FD$, $C'DE$ are all straight lines; and prove that the locus of the centres of conics circumscribing the quadrilateral $A'B'ED$ is

$$c_0(\alpha^2 - \beta^2) + \gamma(a_0\alpha - b_0\beta) = 0,$$

α, β, γ being trilinear coordinates referred to the triangle ABC. (Trinity Hall.)

31. Prove that the conics

$$a_0\sqrt{\alpha \tan\theta} + b_0\sqrt{\beta\tan\phi} + c_0\sqrt{\gamma\tan\psi} = 0,$$

$$a_0\sqrt{\alpha\cot\theta} + b_0\sqrt{\beta\cot\phi} + c_0\sqrt{\gamma\cot\psi} = 0$$

are parabolas if

$$a_0\tan\theta + b_0\tan\phi + c_0\tan\psi = 0$$

and

$$a_0\cot\theta + b_0\cot\phi + c_0\cot\psi = 0,$$

the coordinates being trilinear. (Pembroke.)

32. Tangents, at the centres of the inscribed and escribed circles of a triangle, to the rectangular hyperbola passing through these centres meet in pairs on the sides of the triangle. (Clare.)

33. Prove that the trilinear (or areal) coordinates of any four points may be put in the form $(\pm f, \pm g, \pm h)$. Find the equation of the rectangular hyperbola through these four points. (King's etc.)

34. Shew that the centre of any conic passing through the angular points of the triangle of reference and the centre of the inscribed circle lies on the conic whose equation is

$$a_0\alpha^2 + b_0\beta^2 + c_0\gamma^2 - (b_0 + c_0)\beta\gamma - (c_0 + a_0)\gamma\alpha - (a_0 + b_0)\alpha\beta = 0,$$

the coordinates being trilinear. (Corpus etc.)

35. From a fixed point two secants are so drawn to a fixed conic that the four points of intersection lie on a circle. Shew that the locus of the centre of the circle is a straight line and determine its position. (Trinity.)

36. If the equation of any conic referred to trilinear coordinates be

$$\frac{\alpha^2}{u} + \frac{\beta^2}{v} + \frac{\gamma^2}{w} = 0,$$

shew that the equation to its director circle is

$$\alpha^2(v + w) + \beta^2(w + u) + \gamma^2(u + v)$$
$$+ 2u\beta\gamma \cos A_0 + 2v\gamma\alpha \cos B_0 + 2w\alpha\beta \cos C_0 = 0.$$

(Clare etc.)

37. Prove that the diameter of the conic

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0,$$

which bisects chords parallel to

$$l\alpha + m\beta + n\gamma = 0,$$

is $\quad u\alpha(mc_0 - nb_0) + v\beta(na_0 - lc_0) + w\gamma(lb_0 - ma_0) = 0$

in trilinear coordinates.

38. Shew that the common chord of the conic

$$yz + zx + xy = 0$$

and its circle of curvature at the angular point A of the triangle of reference has for its equation

$$y(a_0^2 - c_0^2) + z(a_0^2 - b_0^2) = 0,$$

the coordinates being areals. (Corpus etc.)

39. If the tangent at A to a circum-conic of the triangle ABC meet BC produced in K, find the equation of the conic which touches AC and AK at their middle points and also touches CK: and shew that the polar of K with respect to this conic is

$$a_0 b_0 \alpha + (2c_0^2 - b_0^2)\beta + b_0 c_0 \gamma = 0$$

in the case when the circumscribing conic is a circle, the coordinates being trilinear. (Pembroke.)

40. The angular points A, B, C of a triangle are joined to any point O, and OA, OB, OC meet the opposite sides in α, β, γ. Shew that if the conic through $\alpha\beta\gamma$ and the middle points of the sides of the triangle ABC be a rectangular hyperbola, then O lies on the circle round ABC. (Clare etc.)

41. Assuming that the centroid, the orthocentre, the centres of the circum-circle and the nine-points circle of the triangle of reference ABC, lie on a straight line, prove that its areal equation is

$$\frac{b_0^2 - c_0^2}{a_0} \cos A_0 x + \frac{c_0^2 - a_0^2}{b_0} \cos B_0 y + \frac{a_0^2 - b_0^2}{c_0} \cos C_0 z = 0.$$

Shew also in any way that this straight line passes through the four corresponding points for the triangle DEF, where D, E, F are the middle points of the sides of the triangle ABC. (Peterhouse etc.)

42. Three conics have respectively as foci the points B and C, C and A, A and B. Prove that the point of intersection of the two common tangents of any two of the conics lie upon a side of the triangle ABC and that the three such points of intersection are collinear.

(Clare etc.)

43. Shew that the trilinear equation to the ellipse through B and C which has one focus at the angular point A of the triangle of reference ABC and the other focus in BC is

$$\alpha^2 \sin^2 \frac{A_0}{2} + \beta\gamma + \gamma\alpha + \alpha\beta = 0.$$

(Jesus etc.)

44. Shew that if the coordinates be trilinear the two lines

$$\{b_0(wu' - w'u)\,\alpha + a_0(vw' - v'w)\,\beta\}\,(a_0\alpha + b_0\beta)$$
$$= (uv' - u'v)\,c_0^2\alpha\beta$$

are parallel to a pair of conjugate diameters in each of the conics

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0,$$
$$u'\alpha^2 + v'\beta^2 + w'\gamma^2 = 0.$$

45. If $ax^2 + by^2 + cz^2 = 0$ be a parabola, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0,$$

and that the equation to its directrix is

$$a(bb_0^2 + cc_0^2)\,x + b(cc_0^2 + aa_0^2)\,y + c(aa_0^2 + bb_0^2)\,z = 0.$$

The centre of the circle circumscribing every triangle, self-conjugate with regard to a parabola, lies on the directrix.

46. Shew that the directrix of the parabola which touches the four lines $(l, \pm m, \pm n)$ is in trilinear coordinates

$$\frac{\alpha}{a_0}(\overline{b_0^2 - c_0^2}\, l^2 - a_0^2\, \overline{m^2 - n^2}) + \text{similar terms} = 0.$$

47. The orthocentre of any triangle circumscribing a parabola lies on the directrix.

48. The locus of the foci of all parabolas self-conjugate with respect to a given triangle is the Nine-Points Circle.

49. Prove that the directrices of all parabolas touching three given lines are concurrent.

50. Through any point on the polar of P with respect to a rectangular hyperbola two chords are drawn, each subtending a right angle at P. Prove that the chords are at right angles. (Clare etc.)

51. Shew that all conics, which have a given direction of axes and to which the triangle of reference is self-conjugate, pass through four fixed concyclic points. (Jesus etc.)

52. The director circles of all conics touching four straight lines have a common radical axis which is the directrix of the parabola of the system.

53. Shew that the point of intersection of the two common tangents of a conic and an osculating circle lies on the confocal conic which passes through the point of osculation.

54. If the foci of a conic inscribed in a triangle lie on a rectangular hyperbola through the vertices of the triangle, prove that the line joining the foci passes through the circumcentre. (Math. Tripos.)

55. Each of three hyperbolas touches the other two and each has for its asymptotes two of the sides of a given triangle; prove that the three tangents drawn at the points of contact of the hyperbolas meet in a point. (Peterhouse.)

56. Prove that the locus of the focus of a parabola touching two fixed straight lines and whose axis passes through a fixed point is a rectangular hyperbola. (Trinity.)

57. A', B', C' are the middle points of the sides BC, CA, AB of a triangle and $A''B''C''$ is the triangle formed by the tangents $B''AC''$, $C''BA''$, $A''CB''$ to the circle circumscribing ABC. Shew that the six points of intersection of the non-corresponding sides of the triangles $A'B'C'$, $A''B''C''$ lie on a conic. (Trinity.)

58. From a fixed point a chord PQ is drawn to a parabola and a circle is described on PQ as diameter meeting the curve again in two new points P', Q'. Shew that the chord $P'Q'$ passes through a fixed point. (Trinity.)

59. If a quadrilateral be inscribed in a circle, one of the three diagonals of the quadrilateral passes through the focus of the parabola touching its four sides. (Queens'.)

60. Let α, β, γ be trilinear coordinates and let a_0, b_0, c_0 be the lengths of the sides of the triangle of reference. Prove that the coordinates of the foci of the ellipse

$$\sqrt{\frac{\alpha}{a_0}}+\sqrt{\frac{\beta}{b_0}}+\sqrt{\frac{\gamma}{c_0}}=0$$

are respectively proportional to

$$\frac{c_0}{b_0},\ \frac{a_0}{c_0},\ \frac{b_0}{a_0} \text{ and } \frac{b_0}{c_0},\ \frac{c_0}{a_0},\ \frac{a_0}{b_0}.$$

(Trinity.)

61. If the axes of two parabolas, each touching all the sides of a triangle, intersect at right angles, prove that their point of intersection lies on the circumcircle of the triangle. (Jesus etc.)

62. Find all the common tangents of the conics $y^2=4zx$ and $z^2=4xy$ and shew that, if the conics have a common focus, the triangle of reference must be equilateral. (St John's.)

63. From a point P perpendiculars PD, PE, PF are drawn to the sides BC, CA, AB respectively of a triangle ABC. If P has been chosen in such a way that AD, BE, CF meet in a point Q, prove that PQ passes through a fixed point R; and the line joining R to the orthocentre of ABC is bisected by the circum-centre. (St John's.)

64. If α, β, γ are ordinary trilinear coordinates and λ, μ, ν the corresponding tangential coordinates, prove that

$$\beta\gamma+\gamma\alpha+\alpha\beta=0,$$

$$(1-\cos A_0)\,\mu\nu+(1-\cos B_0)\,\nu\lambda+(1-\cos C_0)\,\lambda\mu=0$$

are confocal; and that in all there are just four such pairs of confocals for which the triangle of reference is a circum-inscribed triangle. (St John's.)

65. Shew that, if the triangle of reference be equilateral, the locus of the foci of conics which touch the four lines

$$lx \pm my \pm nz = 0$$

is (in either areal or trilinear coordinates)

$$2(l^2+m^2+n^2)\,xyz$$
$$= l^2x^2(x-y-z)+m^2y^2(y-z-x)+n^2z^2(z-x-y).$$

(Clare etc.)

66. The three perpendiculars of a triangle are divided in any given ratio λ, and from the three points of division perpendiculars are drawn on the non-corresponding sides. Prove that the six feet of these perpendiculars lie on a circle. Shew also that as λ varies, this circle envelopes a conic having double contact with the circum-circle of the triangle, and that the centre of the circle moves on a fixed straight line. (Clare etc.)

67. A rectangular hyperbola circumscribes a triangle ABC, and passes through one focus of a conic inscribed in the triangle. Prove that the centre of the rectangular hyperbola lies on the auxiliary circle of the inscribed conic. (St John's.)

68. If ABC be the triangle of reference in trilinear coordinates, and PD, PE, PF the perpendiculars from P on the sides BC, CA, AB respectively, find the coordinates of D, E, F in terms of the coordinates of P, and find the locus of P when AD, BE, CF are concurrent.

If D, E, F lie on $l\alpha+m\beta+n\gamma=0$, then shew that

$$\frac{l}{l-m\cos C_0-n\cos B_0}+\frac{m}{m-n\cos A_0-l\cos C_0}$$
$$+\frac{n}{n-l\cos B_0-m\cos A_0}-1=0,$$

and interpret this result. (St John's.)

CHAPTER V

PARAMETRIC REPRESENTATION

1. *Homogeneous Coordinates.*

In Homogeneous Coordinates we deal with Coordinates which are respectively known constant multiples of the Areal Coordinates.

Thus if (x, y, z) be the "actual areal coordinates" and (ξ, η, ζ) some form of Homogeneous Coordinates, then we can transform from the one system to the other by means of the following formulae:

$$\xi = px,$$

$$\eta = qy,$$

$$\zeta = rz,$$

where p, q, r are known constant multipliers and thus define the system of Homogeneous Coordinates under consideration.

N.B. Areal and Trilinear Coordinates are merely particular cases of Homogeneous Coordinates.

The following propositions will be evident:

I. The identical relation connecting ξ, η, ζ in the case of the above system of Homogeneous Coordinates is

$$\frac{\xi}{p} + \frac{\eta}{q} + \frac{\zeta}{r} = 1.$$

II. The equation of the Line at Infinity is

$$\frac{\xi}{p}+\frac{\eta}{q}+\frac{\zeta}{r}=0.$$

III. *The equation of a straight line is of the first degree in the homogeneous coordinates.*

IV. *The equation of a conic is of the second degree in the homogeneous coordinates.*

V. Tangential coordinates are transformed thus:

The line $$lx+my+nz=0$$

in areal coordinates becomes

$$\frac{l}{p}\xi+\frac{m}{q}\eta+\frac{n}{r}\zeta=0$$

in homogeneous coordinates.

Hence if we denote the tangential coordinates of a line in homogeneous coordinates by λ, μ, ν we see that

$$\lambda:\mu:\nu=\frac{l}{p}:\frac{m}{q}:\frac{n}{r}.$$

Thus the equation of a point is of the first degree in λ, μ, ν in homogeneous tangential coordinates.

The equation of a conic is of the second degree in λ, μ, ν in homogeneous tangential coordinates.

VI. *The formulae for tangents, poles and polars are of the same form in homogeneous coordinates as in areal coordinates.*

VII. *The pairs of lines*

$$\alpha\xi^2+2\gamma\xi\eta+\beta\eta^2=0$$

and $$\alpha'\xi^2+2\gamma'\xi\eta+\beta'\eta^2=0$$

will harmonically separate each other if

$$\alpha\beta'+\alpha'\beta=2\gamma\gamma'.$$

For changing into areal coordinates the lines become

$$\alpha p^2x^2 + 2\gamma pqxy + \beta q^2y^2 = 0,$$

$$\alpha' p^2x^2 + 2\gamma' pqxy + \beta' q^2y^2 = 0,$$

and these will harmonically separate each other if

$$\alpha\beta' p^2q^2 + \alpha'\beta p^2q^2 = 2\gamma\gamma' p^2q^2,$$

i.e. if $$\alpha\beta' + \alpha'\beta = 2\gamma\gamma'.$$

2. *Complete Quadrangle.*

Canonical Form for four points in Homogeneous Coordinates.

We have seen (Chap. I, Art. 18) that the Canonical form for four points in Areal Coordinates is

$$(x', y', z'); (-x', y', z'); (x', -y', z'); (x', y', -z').$$

Take as homogeneous coordinates the system in which

$$\xi = \frac{x}{x'}, \quad \eta = \frac{y}{y'}, \quad \zeta = \frac{z}{z'}$$

and the four points become

$$(1, 1, 1); (-1, 1, 1); (1, -1, 1); (1, 1, -1).$$

3. *Complete Quadrilateral.*

Canonical Form for four lines in Homogeneous Coordinates.

By Chap. I, Art. 19, we can in Areal Coordinates take four lines in the form

$$lx + my + nz = 0,$$

$$-lx + my + nz = 0,$$

$$lx - my + nz = 0,$$

$$lx + my - nz = 0.$$

Choose the system of Homogeneous Coordinates in which $\xi = lx$, $\eta = my$, $\zeta = nz$ and our lines become

$$\xi + \eta + \zeta = 0,$$
$$-\xi + \eta + \zeta = 0,$$
$$\xi - \eta + \zeta = 0,$$
$$\xi + \eta - \zeta = 0.$$

Hereafter we shall use (x, y, z) to denote Homogeneous Coordinates unless otherwise stated.

4. Consider the formulae

$$x = f(t),$$
$$y = \phi(t),$$
$$z = \psi(t).$$

If we assign various values to t, we shall get a series of values for x, y, z and these we can represent by points with respect to some triangle of reference. If we let t vary over all values, the point (x, y, z) will describe some curve. If f, ϕ, ψ denote algebraic polynomials in t, then to every value of t will correspond one and only one value for each of x, y and z respectively. Hence corresponding to an assigned value for t there will be one and only one position of the point (x, y, z) on the curve, and we may therefore briefly refer to the Point $\equiv t$.

t is called the Parameter of the point.

5. *To find the Parametric Equations to a Conic.*

Let P and Q be two points on the conic and let the tangents at these points meet in O.

Then the equation to the conic will, by Chap. II, Art. 12, Case II, be of the form

$$(l_0x + m_0y + n_0z)^2 = (l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z) \ldots (1),$$

where $l_0x + m_0y + n_0z = 0$ is the equation to PQ,

$l_1x + m_1y + n_1z = 0$ „ „ OP,

$l_2x + m_2y + n_2z = 0$ „ „ OQ.

We may write (1) thus

$$\frac{l_0x + m_0y + n_0z}{l_1x + m_1y + n_1z} = \frac{l_2x + m_2y + n_2z}{l_0x + m_0y + n_0z} = t.$$

Hence

$$\frac{l_0x + m_0y + n_0z}{t} = \frac{l_1x + m_1y + n_1z}{1} = \frac{l_2x + m_2y + n_2z}{t^2} = k \ldots (2).$$

$$\therefore \quad l_0x + m_0y + n_0z = kt,$$
$$l_1x + m_1y + n_1z = k,$$
$$l_2x + m_2y + n_2z = kt^2,$$

and on solving we get

$$x = a_0t^2 + 2a_1t + a_2,$$
$$y = b_0t^2 + 2b_1t + b_2,$$
$$z = c_0t^2 + 2c_1t + c_2.$$

Hence every conic can be thus represented.

6. *Conversely the set of equations*

$$x = a_0t^2 + 2a_1t + a_2$$
$$y = b_0t^2 + 2b_1t + b_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$z = c_0t^2 + 2c_1t + c_2$$

represents a conic.

For this curve meets any straight line

$$lx + my + nz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in values of t given by

$$l(a_0t^2 + 2a_1t + a_2) + m(b_0t^2 + 2b_1t + b_2)$$
$$+ n(c_0t^2 + 2c_1t + c_2) = 0 \ldots (3),$$

that is in two points, since (3) is a quadratic.

COROLLARY.

$$x = t^2,$$
$$y = 2t,$$
$$z = 1$$

is the conic $y^2 = 4xz$ which touches BA at A and BC at C.

7. *To find the points in which the conic*

$$\begin{aligned} x &= a_0t^2 + 2a_1t + a_2 \\ y &= b_0t^2 + 2b_1t + b_2 \quad \text{..................(1)} \\ z &= c_0t^2 + 2c_1t + c_2 \end{aligned}$$

cuts the curve $F(x, y, z) = 0$ ....(2).

If we substitute from (1) in (2), we get the points whose parameters are given by

$$F(a_0t^2 + 2a_1t + a_2,\ b_0t^2 + 2b_1t + b_2,\ c_0t^2 + 2c_1t + c_2) = 0.$$

8. *To find geometrical interpretations for the roots of the three quadratic functions of t, in*

$$x = a_0t^2 + 2a_1t + a_2,$$
$$y = b_0t^2 + 2b_1t + b_2,$$
$$z = c_0t^2 + 2c_1t + c_2.$$

This conic will cut BC, i.e. $x = 0$, in points whose parameters are given by

$$a_0t^2 + 2a_1t + a_2 = 0,$$

and so for the others.

COROLLARY.

In the case of the conic

$$x = ut + v,$$
$$y = b_0t^2 + 2b_1t + b_2,$$
$$z = c_0t^2 + 2c_1t + c_2,$$

the curve cuts BC in points whose parameters are $-\frac{v}{u}$ and ∞.

9. *To find how the curve is situated relative to the triangle of reference.*

Find the parameters of the points in which the conic cuts the three sides of the triangle of reference and thence calculate the point coordinates of the points of intersection. We shall thus in general get the coordinates of *six* points on the curve very conveniently and only *five* are necessary to uniquely define the conic.

10. Example.

To draw the conic

$$x = t(t-2),$$
$$y = t(4-t),$$
$$z = 1.$$

Draw the conic cutting the sides of the triangle of reference in the most general way and then insert at the points of intersection the values of the parameters (Fig. 34). We thus get the following diagram:

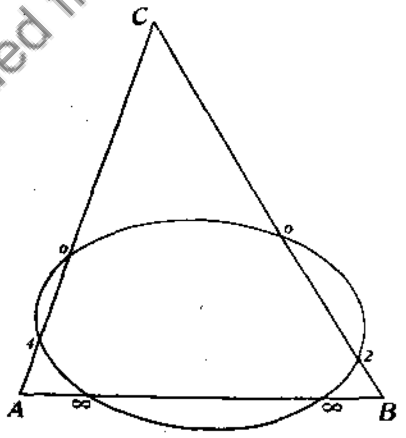

Fig. 34.

Hence plainly the curve must take up the following position (Fig. 35).

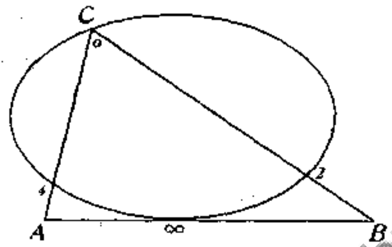

Fig. 35.

11. *To find the equation to the chord joining the points t_1 and t_2.*

The line will be

$$\begin{vmatrix} x & y & z \\ a_0t_1^2+2a_1t_1+a_2 & b_0t_1^2+2b_1t_1+b_2 & c_0t_1^2+2c_1t_1+c_2 \\ a_0t_2^2+2a_1t_2+a_2 & b_0t_2^2+2b_1t_2+b_2 & c_0t_2^2+2c_1t_2+c_2 \end{vmatrix}=0,$$

which, on expanding and dividing out the factor t_1-t_2, we get in the form

$$\{2(b_0c_1-b_1c_0)t_1t_2+(b_0c_2-b_2c_0)\overline{t_1+t_2}+2(b_1c_2-b_2c_1)\}x$$
$$+\{2(c_0a_1-c_1a_0)t_1t_2+(c_0a_2-c_2a_0)\overline{t_1+t_2}+2(c_1a_2-c_2a_1)\}y$$
$$+\{2(a_0b_1-a_1b_0)t_1t_2+(a_0b_2-a_2b_0)\overline{t_1+t_2}+2(a_1b_2-a_2b_1)\}z=0.$$

12. *To find the tangent at the point t.*

Put $t_1=t_2=t$ in the formula of Art. 11 and we get

$$x(\overline{b_0c_1-b_1c_0}\,t^2+\overline{b_0c_2-b_2c_0}\,t+\overline{b_1c_2-b_2c_1})$$
$$+y(\overline{c_0a_1-c_1a_0}\,t^2+\overline{c_0a_2-c_2a_0}\,t+\overline{c_1a_2-c_2a_1})$$
$$+z(\overline{a_0b_1-a_1b_0}\,t^2+\overline{a_0b_2-a_2b_0}\,t+\overline{a_1b_2-a_2b_1})=0.$$

COROLLARY I.

The line coordinates l; m; n of the tangent to a conic can be expressed as quadratic functions of the parameter t of the point of contact.

COROLLARY II.

If we render homogeneous the expressions for x, y, z in terms of t by means of

$$t' = 1 \; (e.g.\ x = a_0t^2 + 2a_1tt' + a_2t'^2),$$

the above equation to the tangent may be more easily remembered in the form

$$\begin{vmatrix} x & y & z \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \\ \dfrac{\partial x}{\partial t'} & \dfrac{\partial y}{\partial t'} & \dfrac{\partial z}{\partial t'} \end{vmatrix} = 0.$$

13. *To find the pole of the chord joining the points t_1 and t_2.*

The tangent at the point t_1 is seen from last article to be

$$\begin{vmatrix} x & y & z \\ a_0t_1 + a_1 & b_0t_1 + b_1 & c_0t_1 + c_1 \\ a_1t_1 + a_2 & b_1t_1 + b_2 & c_1t_1 + c_2 \end{vmatrix} = 0 \quad \ldots(1).$$

Hence the point

$$(a_0t_1t_2 + a_1\overline{t_1 + t_2} + a_2,\ b_0t_1t_2 + b_1\overline{t_1 + t_2} + b_2,\ c_0t_1t_2 + c_1\overline{t_1 + t_2} + c_2) \quad \ldots\ldots\ldots\ldots(2),$$

or as it may be written

$$(\overline{a_0t_1 + a_1}t_2 + \overline{a_1t_1 + a_2},\ \overline{b_0t_1 + b_1}t_2 + \overline{b_1t_1 + b_2},\ \overline{c_0t_1 + c_1}t_2 + \overline{c_1t_1 + c_2}) \quad \ldots\ldots\ldots\ldots(3),$$

plainly lies on the line (1) as may be seen if we substitute from (3) in (1) and note that then the top row is obtained by multiplying the constituents of the second row by t_2 and adding to the corresponding ones of the third row. The determinant will thus identically vanish.

Hence the point (2) lies on the tangent at the point t_1.

Similarly the point (2) lies on the tangent at the point t_2.

Hence the point (2) must be the pole of the line joining the points t_1 and t_2.

N.B. *To write down instantaneously the pole of the line joining t_1 and t_2, in the general parametric expressions for x, y, z change t^2 into t_1t_2 and $2t$ into t_1+t_2.*

14. *If a triangle of reference be assigned, we can so choose our formulae of parametric representation that three assigned points on a conic shall have three assigned parametric values.*

Let the given conic be parametrically represented so that

$$P \equiv (a_0t^2+2a_1t+a_2,\ b_0t^2+2b_1t+b_2,\ c_0t^2+2c_1t+c_2) \ldots(1)$$

is any point on it. Let $P_1 \equiv t_1$, $P_2 \equiv t_2$, $P_3 \equiv t_3$ be the three assigned points which we wish to be represented by the parametric values τ_1, τ_2, τ_3 respectively.

Let $$\frac{\tau-\tau_1}{\tau-\tau_2}=\lambda\frac{t-t_1}{t-t_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then plainly when

$$t=t_1, \therefore \tau=\tau_1; \text{ and when } t=t_2, \therefore \tau=\tau_2.$$

Choose λ to satisfy

$$\frac{\tau_3-\tau_1}{\tau_3-\tau_2}=\lambda\frac{t_3-t_1}{t_3-t_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

giving $$\frac{\tau-\tau_1}{\tau-\tau_2}\cdot\frac{t_3-t_1}{t_3-t_2}=\frac{t-t_1}{t-t_2}\cdot\frac{\tau_3-\tau_1}{\tau_3-\tau_2} \quad \ldots\ldots\ldots(4).$$

Hence (4) gives t in terms of τ in the form

$$t=\frac{p\tau+q}{r\tau+s}.$$

Hence substituting in (1) and clearing of fractions we see that P can be represented by means of quadratic functions of τ such that when

$$t=t_1,\ t_2,\ t_3,$$
$$\tau=\tau_1,\ \tau_2,\ \tau_3,$$

where the points P_1, P_2, P_3 were previously assigned and the three parameters τ_1, τ_2, τ_3 respectively representing them were also previously assigned.

15. *To find the parametric equations to a Circum-Conic.*

In virtue of Art. 14 let us assign to the points A, B, C the parameters 0, ∞, -1 respectively. Then, by Art. 8, $x=a(t+1)$, $y=bt(t+1)$, $z=ct$, or using the homogeneous coordinates in which $x=a\xi$, $y=b\eta$, $z=c\zeta$, we get

$$\xi=(t+1),$$
$$\eta=t(t+1),$$
$$\zeta=t.$$

16. *To find the parametric equations to an inscribed conic.*

Let the conic touch BC at D, CA at E and AB at F, and let the parameters of D, E, F be 0, ∞, -1 respectively.

Then reasoning as in last article we get

$$x=t^2,$$
$$y=1,$$
$$z=(t+1)^2.$$

17. *To find the parametric equations to a conic with respect to which the triangle of reference is self-conjugate.*

Using suitable homogeneous coordinates, let the conic be

$$x^2 + y^2 = z^2.$$

Then
$$\frac{x}{\cos\psi} = \frac{y}{\sin\psi} = \frac{z}{1}$$

will satisfy this conic,

i.e.
$$\frac{x}{\cos^2\frac{\psi}{2} - \sin^2\frac{\psi}{2}} = \frac{y}{2\sin\frac{\psi}{2}\cos\frac{\psi}{2}} = \frac{z}{\cos^2\frac{\psi}{2} + \sin^2\frac{\psi}{2}},$$

i.e. if $t = \tan\frac{\psi}{2}$, we may take

$$x = 1 - t^2,$$
$$y = 2t,$$
$$z = 1 + t^2.$$

18. *To find the parametric equations to a conic touching CA at A and CB at B.*

Let the parameters of A and B be 0 and ∞ respectively.

$\therefore$
$$x = 1,$$
$$y = t^2,$$
$$z = 2t.$$

19. *Tangential parametric equations.*

If we take the conic

$$l = A_0t^2 + 2A_1t + A_2,$$
$$m = B_0t^2 + 2B_1t + B_2,$$
$$n = C_0t^2 + 2C_1t + C_2,$$

we see at once that

$$A_0 t^2 + 2A_1 t + A_2 = 0$$

gives the parameters of the points of contact of the two tangents from A to the conic.

20. *To find how the Curve is situated relatively to the triangle of reference.*

Find the parameters of the points of contact of the tangents drawn to the conic from the vertices of the triangle of reference and thence calculate the line coordinates of these tangents. We shall thus in general get the line coordinates of *six* tangents to the conic and only five are necessary to uniquely define the conic.

21. Example.

To draw the conic

$$l = t^2,$$
$$m = t + 1,$$
$$n = t(t + 1).$$

Draw the figure as for the general case and insert the values of the parameters of the points of contact of tangents drawn from A, B, C respectively.

We thus get (Fig. 36)

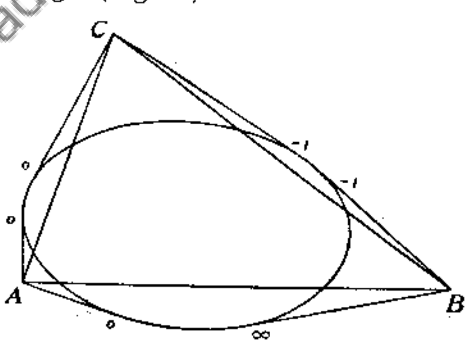

Fig. 36.

All these parametric values can only be satisfied by the conic taking up a position thus (Fig. 37):

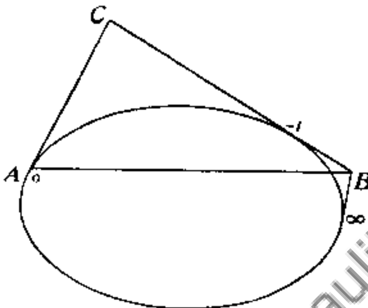

Fig. 37.

22. *To find the tangential parametric equation to a Circum-Conic.*

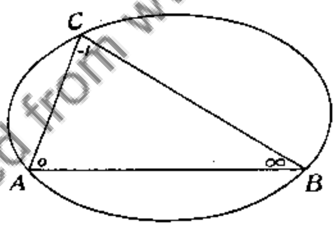

Fig. 38.

Let the parameters of A, B, C be $0, \infty, -1$ respectively (Fig. 38).

Hence we see from Chap. v, Art. 19, that

$$l = t^2,$$

$$m = 1,$$

$$n = (t+1)^2,$$

if we choose suitable Homogeneous Coordinates.

23. *To find the tangential parametric equation to an Inscribed Conic.*

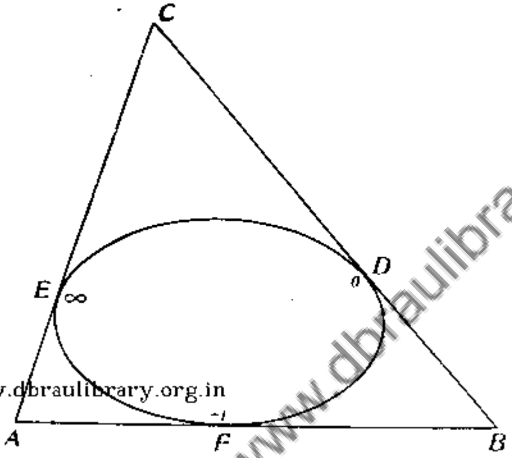

Fig. 39.

Let the points of contact of the conic with the sides of the triangle of reference be D, E, F, and let their parameters be 0, ∞, -1 respectively (Fig. 39).

Then
$$l = t + 1,$$
$$m = t(t+1),$$
$$n = t.$$

24. *To find the tangential parametric equation to a conic with respect to which the triangle of reference is self-conjugate.*

Taking the conic in the form

$$l^2 + m^2 = n^2,$$

we may prove exactly as in Chap. V, Art. 17, that the line coordinates may be parametrically represented thus

$$l = 1 - t^2,$$
$$m = 2t,$$
$$n = 1 + t^2.$$

25. *To find the tangential parametric equation to a conic touching CA at A and CB at B.*

Taking the parameters of A and B as ∞ and 0 respectively, we get

$$l = 1,$$
$$m = t^2,$$
$$n = 2t.$$

26. EXAMPLE 1.

The reciprocal of a conic is a conic.

Take the conic to be reciprocated as

$$x = 1$$
$$y = t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$z = 2t$$

and let the conic of reciprocation be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots(2).$$

Then the polar of the point $P \equiv t$ on the conic (1) with respect to the conic (2) is

$$(ax + hy + gz) + (hx + by + fz)\,t^2 + 2\,(gx + fy + cz)\,t = 0,$$

which envelopes the conic

$$(gx + fy + cz)^2 = (ax + hy + gz)(hx + by + fz) = 0.$$

EXAMPLE 2.

A conic touches CA at A and CB at B. S is a variable point on the conic and the tangent at S meets CB in Q and CA in R. Prove that the locus of the intersection of AQ and BR is a conic.

Take the conic in the form

$$x = 1, \quad y = t^2, \quad z = 2t.$$

The tangent at t will be

$$xt^2 + y - zt = 0.$$

The equation to AQ will be, therefore,

$$y - zt = 0.$$

The equation to BR will be, therefore,

$$xt - z = 0.$$

Hence the locus of the intersection of AQ and BR will be

$$xy = z^2.$$

EXAMPLE 3.

Two conics touch at A and B. P is a variable point on the first. PA meets the second in Q and PB meets the second in R. Prove that QR envelopes a conic.

Let the first conic be

$$kz^2 + 2xy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let the second conic be

$$x = 1, \; y = t^2, \; z = 2t \quad \ldots\ldots\ldots\ldots\ldots(2).$$

Let $\qquad Q \equiv t_1$ and $R \equiv t_2$.

Hence the line coordinates of the line QR will be given by

$$(2t_1t_2,\ 2,\ -\overline{t_1 + t_2}) \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now the equation to AQ is

$$2y = t_1 z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and the equation to BR is

$$2t_2 x = z,$$

Hence the point

$$P \equiv (1,\ t_1t_2,\ 2t_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But P lies on the conic

$$kz^2+2xy=0.$$

Hence $$2kt_2+t_1=0 \quad \text{..................(6).}$$

Thus the line coordinates of QR will be from (3)

$$l=-4kt_2^2,$$
$$m=2,$$
$$n=(2k-1)\,t_2,$$

which is a conic touching the others at A and B also.

27. *To prove that the cross-ratio of the four rays*

$$x=py,$$
$$x=qy,$$
$$x=ry,$$
$$x=sy,$$

passing through C, is $=[pqrs]$.

Let these rays cut AB in the points P, Q, R, S respectively.

Then, as proved in Chap. I, Art. 17,

$$AP=\frac{c_0}{p+1};\ AQ=\frac{c_0}{q+1};\ AR=\frac{c_0}{r+1};\ AS=\frac{c_0}{s+1}\text{...(1).}$$

Hence

$$[PQRS]=\frac{PQ\,.\,RS}{PS\,.\,RQ}=\frac{(AQ-AP)(AS-AR)}{(AS-AP)(AQ-AR)}$$

$$=\frac{\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\left(\frac{1}{s+1}-\frac{1}{r+1}\right)}{\left(\frac{1}{s+1}-\frac{1}{p+1}\right)\left(\frac{1}{q+1}-\frac{1}{r+1}\right)}$$

$$=\frac{(p-q)(r-s)}{(p-s)(r-q)} \quad \text{.............................(2)}$$

$$=[pqrs] \quad \text{.................................(3).}$$

28. *If the two lines*

$$L_1 \equiv l_1x + m_1y + n_1z = 0,$$

$$L_2 \equiv l_2x + m_2y + n_2z = 0$$

meet in O and if $L_1 + pL_2$, $L_1 + qL_2$, $L_1 + rL_2$, $L_1 + sL_2$ *be four lines through their point of intersection, then the cross-ratio of these four lines will be equal to* $[pqrs]$.

Let the given four lines meet AB in the points P, Q, R, S respectively.

The equation to CP will be

$$(l_1 + pl_2)\,x + (m_1 + pm_2)\,y = 0,$$

which may be written in the form

$$x = -\frac{m_1 + pm_2}{l_1 + pl_2}\,y,$$

and so for the others.

Hence the cross-ratio of the given pencil of four lines

$$=[PQRS]$$

$$=\left[-\frac{m_1+pm_2}{l_1+pl_2},\ -\frac{m_1+qm_2}{l_1+ql_2},\ -\frac{m_1+rm_2}{l_1+rl_2},\ -\frac{m_1+sm_2}{l_1+sl_2}\right]$$

(by last article)

$$=\frac{\left(-\dfrac{m_1+pm_2}{l_1+pl_2}+\dfrac{m_1+qm_2}{l_1+ql_2}\right)\left(-\dfrac{m_1+rm_2}{l_1+rl_2}+\dfrac{m_1+sm_2}{l_1+sl_2}\right)}{\left(-\dfrac{m_1+pm_2}{l_1+pl_2}+\dfrac{m_1+sm_2}{l_1+sl_2}\right)\left(-\dfrac{m_1+rm_2}{l_1+rl_2}+\dfrac{m_1+qm_2}{l_1+ql_2}\right)}$$

$$=\frac{(p-q)(r-s)}{(p-s)(r-q)}$$

$$=[pqrs],$$

as can be easily found if we multiply out and cancel.

29. Cross-ratio Property of a Conic. If P_1, P_2, P_3, P_4 be four fixed points on a conic and O a variable point also on the conic, then the cross-ratio of the pencil $O[P_1P_2P_3P_4]$ is constant for all positions of O on the conic [29].

The constant cross-ratio subtended by the four points t_1, t_2, t_3, t_4 at any point on the conic

$$x = a_0t^2 + 2a_1t + a_2,$$
$$y = b_0t^2 + 2b_1t + b_2,$$
$$z = c_0t^2 + 2c_1t + c_2$$

is equal to the cross-ratio of their parameters, i.e. $= [t_1t_2t_3t_4]$.

Since the cross-ratio subtended by four points on a conic at any point on the conic is constant, we may consider the cross-ratio subtended by t_1, t_2, t_3, t_4 at the point $t \equiv 0$.

Let the point $t \equiv 0$ be O, $P_1 \equiv t_1$ etc.

Then the equation to OP_1 is (Chap. v, Art. 11)

$$\begin{vmatrix} x & y & z \\ a_0t_1^2 + 2a_1t_1 + a_2 & b_0t_1^2 + 2b_1t_1 + b_2 & c_0t_1^2 + 2c_1t_1 + c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0,$$

i.e. subtracting the third row from the second and cancelling t_1

$$\begin{vmatrix} x & y & z \\ a_0t_1 + 2a_1 & b_0t_1 + 2b_1 & c_0t_1 + 2c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Hence the equation to OP_1 will be

$$t_1\{(b_0c_2 - b_2c_0)x + (c_0a_2 - c_2a_0)y + (a_0b_2 - a_2b_0)z\}$$
$$= 2\{(b_2c_1 - b_1c_2)x + (c_2a_1 - c_1a_2)y + (a_2b_1 - a_1b_2)z\},$$

i.e. $$t_1L_1 = 2L_2,$$

Since only even powers of t occur, the roots of (4) must be of the form $t_1, -t_1, t_2, -t_2$ and since the product of the roots $= +1$ we may take the roots as

$$t, -t, \frac{1}{t}, -\frac{1}{t}.$$

Hence the cross-ratio ρ required will be given by

$$\rho = \left[t, -t, \frac{1}{t}, -\frac{1}{t}\right]$$

$$= \frac{4t^2}{(1+t^2)^2}$$

$$= \frac{a^2 - c^2}{a^2 - b^2} \quad \text{by (4).}$$

EXAMPLES. V.

1. Prove that the equations

$$x = at + b,$$
$$y = ct + d,$$
$$z = et + f$$

represent parametrically a straight line.

2. Prove that with a given triangle of reference a homogeneous system of coordinates can always be chosen so that any given line can be represented in the form

$$x = t,$$
$$y = 1,$$
$$z = -(t+1).$$

3. Prove that, in the case of the conic

$$x = 1,$$
$$y = t^2,$$
$$z = 2t,$$

the equation to the chord joining the points t_1 and t_2 will be

$$2t_1t_2x + 2y - (t_1 + t_2)z = 0.$$

4. Prove that the tangential parametric equation to

$$x = 1,$$
$$y = t^2,$$
$$z = 2t$$

will be

$$l = t^2,$$
$$m = 1,$$
$$n = -t.$$

5. Prove that the equation to the chord joining the points t_1 and t_2 on the conic

$$x = t + 1,$$
$$y = t(t+1),$$
$$z = t$$

is

$$t_1 t_2 x + y - (t_1 + 1)(t_2 + 1) z = 0.$$

6. Prove that the parametric equation in tangential coordinates to the conic of Question 5 is

$$l = t^2,$$
$$m = 1,$$
$$n = -(1 + t)^2.$$

7. Shew that in the conic $x = t^2$, $y = 1$, $z = 2t$, chords joining points t_1 and t_2, where t_1 and t_2 are connected by the relation

$$a t_1 t_2 + b t_1 + c t_2 + d = 0,$$

envelope the conic

$$lm(b - c)^2 + (dl + am - 2cn)(dl + am - 2bn) = 0.$$

8. Prove that chords joining the points t_1 and t_2, where t_1 and t_2 are connected by the relation

$$\alpha t_1 t_2 + \beta(t_1 + t_2) + \gamma = 0,$$

on the conic
$$x = t + 1,$$
$$y = t(t+1),$$
$$z = t,$$

all pass through the point

$$(\beta - \alpha, \beta - \gamma, \beta).$$

9. Prove that the chord of $x = t^2$, $y = 1$, $z = (t+1)^2$ joining t_1 and t_2 is

$$(t_1 + t_2 + 2)x + (t_1 + t_2 + 2t_1t_2)y - (t_1 + t_2)z = 0.$$

10. Prove that the parametric tangential equation of $x = t^2$, $y = 1$, $z = (t+1)^2$ is $l = t + 1$, $m = t(t+1)$, $n = -t$.

11. A conic touching CA at A and CB at B has the equations $x = 1$, $y = t^2$, $z = 2t$; the point $P \equiv t$ is taken on the conic; AP meets CB in Q and BP meets CA in R. Prove that the tangential coordinates of QR are given by $l = 2t^2$, $m = 2$, $n = -t$.

What is the envelope of QR as P varies?

12. P is a point on the inscribed conic $x = t^2$, $y = 1$, $z = (t+1)^2$, and the tangent at $P \equiv t$ meets CB and CA in Q and R respectively. Prove that the coordinates of the intersection of AQ and BR are $(t, 1, t+1)$. Deduce that the locus of the intersection of AQ and BR is a straight line.

13. $P \equiv t$ is a point on the circum-conic $x = t + 1$, $y = t(t+1)$, $z = t$, and the tangent at P meets CA in Q and CB in R. Prove that the coordinates of the point of intersection of AR and BQ are $\{(t+1)^2, t^2(t+1)^2, t^2\}$.

14. $P \equiv t$ is a point on the circum-conic

$$x = t + 1,$$
$$y = t(t+1),$$
$$z = t.$$

AP meets CB in Q and BP meets CA in R. Shew that the tangential coordinates of the line QR are $(t, 1, -\overline{t+1})$.

15. Shew that in tangential coordinates

$$l = At + B, \quad m = Ct + D, \quad n = Et + F$$

represents a point, and hence interpret the result of Question 14.

16. Prove that if a triangle of reference be assigned, we can by suitably choosing a parameter and a system of homogeneous coordinates represent any given point in the form $l = t,\ m = 1,\ n = -(t+1)$.

17. The conic $x = 1,\ y = t^2,\ z = 2t$ touches CA at A and CB at B. $X \equiv t_1$ and $Y \equiv t_2$ are two fixed points on the conic and $P \equiv t$ is a variable point on the conic. XP meets CB in Q and YP meets CA in R. Prove that the equation to QR is

$$\frac{2tt_2}{t+t_2}x + \frac{2}{t+t_1}y = z.$$

Deduce the envelope of QR.

18. The conic

$$x = 1,$$
$$y = t^2,$$
$$z = 2t$$

touches CA at A and CB at B. $X \equiv t_1$ and $Y \equiv t_2$ are two fixed points on the conic and $P \equiv t$ a variable point on the conic. The tangent at P meets CB in Q and CA in R. Prove that the intersection of XQ and YR is the point

$$\{(t_2 - t)^2, \quad t_2^2(t-t_1)^2, \quad tt_2^2 - t_1(2t-t_1)(2t_2-t)\}.$$

19. $P \equiv (\xi, 1, 0)$ is a point on the base AB of the triangle of reference and CP meets the circum-conic

$$x = t + 1,$$
$$y = t(t + 1),$$
$$z = t$$

in the point $Q \equiv t$. Prove that ξ and t defining P and Q respectively in the ways just indicated satisfy the relation $\xi t = 1$.

20. If $$at^2 + 2ht + b = 0$$
and $$a't^2 + 2h't + b' = 0$$
each define a pair of points on a conic given by parametric equations, prove that

$$(at^2 + 2ht + b) + \lambda(a't^2 + 2h't + b') = 0$$

represents a pair of points on the same conic whose join always passes through a fixed point for all values of λ.

21. A line AP drawn from the vertex of a triangle to a point in the base BC is divided in Q so that $PQ : QA :: BP : PC$. Prove that Q traces out a parabola which passes through B, touches CA at A and has its axis parallel to BC. (Math. Tripos.)

22. If $x : y : z = t^2 + 8t + 2 : 4t^2 + 5t + 1 : 4t^2 - t + 1$, find the tangential equation of the conic and prove that the curve is a parabola if the coordinates are areal.

(St John's.)

23. AB is a fixed chord of a given conic, P is a variable point on the conic. Prove that the locus of the centre of the conic which touches the sides of the triangle APB at their middle points is a conic.

24. A parabola touches two lines at fixed points. Prove that the locus of the middle points of the portions of the tangents to the parabola intercepted by the two given tangents is a straight line.

25. P and Q are any two points on a hyperbola, lines are drawn from P and Q parallel to the asymptotes forming a parallelogram; shew that the other diagonal of this parallelogram passes through the centre of the hyperbola. (Queens'.)

26. If x, y, z be ordinary rectangular Cartesian coordinates, where $z=0$ is the line at infinity, find the condition that the conic

$$x = at^2 + bt + c,$$

$$y = a't^2 + b't + c',$$

$$z = a''t^2 + b''t + c''$$

be a rectangular hyperbola, and interpret geometrically the condition

$$a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c) = 0.$$

(King's.)

27. A square is inscribed in an ellipse, whose semi-axes are a and b, and any point on the ellipse is joined to the corners of the square. Prove that one of the anharmonic ratios of the pencil so formed is $-\dfrac{b^2}{a^2}$.

(Pembroke etc.)

28. Shew that the director-circle of an ellipse cuts the ellipse in four imaginary points. Shew that one of the anharmonic ratios which these four points subtend at any point on the ellipse is $+\frac{b^2}{a^2}$, and that one of the anharmonic ratios that these points subtend on the director-circle is $+\frac{b^4}{a^4}$.

29. The equation of a conic referred to the triangle made by two tangents and the chord of contact is $\alpha\beta = \kappa\gamma^2$. If $\alpha' = 0$ be the tangent at $(\alpha_1, \beta_1, \gamma_1)$, $\beta' = 0$ be the tangent at $(\alpha_2, \beta_2, \gamma_2)$ and $\gamma' = 0$ their chord of contact, shew that the points of intersection of α with α', β with β' and γ with γ' lie on the line

$$\frac{\gamma_2}{\alpha_2}\alpha + \frac{\gamma_1}{\beta_1}\beta = 2\gamma.$$

30. Four points A, B, C, D are taken on

$$\frac{l}{r} = 1 + e\cos\theta,$$

and the corresponding values of θ are $\alpha, \beta, \gamma, \delta$; shew that the anharmonic ratio of the pencil subtended by $[ABCD]$ at any point of the conic is the ratio with the sign changed of some two of the three quantities

$$\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\delta),$$
$$\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\delta),$$
$$\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta).$$ (St John's.)

31. If $ABCD$ be a collinear range, Q the middle point of AC, R the middle point of BD, and P the intersection of the semi-circles on AC, BD as diameters respectively, then shew that the cross-ratio of the range formed with any permutation of the four points A, B, C, D is some one of the six quantities

$$\sin^2\theta,\quad \cos^2\theta,\quad -\tan^2\theta,\quad -\cot^2\theta,\quad \sec^2\theta,\quad \operatorname{cosec}^2\theta,$$

where the angle QPR is 2θ. (St John's.)

32. A, B, C, D are four fixed points on a conic and P is a variable point. Shew that the product of $[PCBD]$ and $[ACPD]$ is constant.

33. Shew that the four points whose eccentric angles are $\alpha, \beta, \gamma, \delta$ subtend at any point on the ellipse a pencil whose anharmonic ratio is

$$\frac{\sin \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\gamma - \delta)}{\sin \frac{1}{2}(\alpha - \delta) \sin \frac{1}{2}(\gamma - \beta)}.$$

(Peterhouse etc.)

34. Shew that in general the equation of any conic can be reduced to the form

$$L(x + ay + bz)^2 + M(y + cz)^2 + Nz^2 = 0,$$

and thence shew that the conic can be represented parametrically as $x = a_1t^2 + 2b_1t + c_1$ etc. (St John's.)

35. Find the equation of the tangent to the parabola $y^2 = 4ax$ (in ordinary rectangular Cartesian coordinates) at the point $a\mu^2, 2a\mu$; and prove that the equation of the other parabola which passes through the origin, and has contact of the second order with the given parabola at the above point, is

$$(4x - 3\mu y)^2 + 4a\mu^2(3x - 2\mu y) = 0.$$

(Pembroke.)

36. Establish the following geometrical construction for a parabola touching two sides of a triangle ABC at B and C. Lines are drawn parallel to the line through A bisecting BC, and the length intercepted between a side and the base is divided into two parts whose ratio is equal to the ratio of the segments of the base, the shorter part of the intercept towards A. The locus of the point is the required parabola. (King's etc.)

37. P is any point on the conic $ax^2+by^2+cz^2=0$ and A, B, C, D are the four points in which it cuts

$$a'x^2+b'y^2+c'z^2=0.$$

Prove that the anharmonic ratio of the pencil $P[ABCD]$ is the ratio of some two of the three quantities

$$\frac{a'}{a}-\frac{b'}{b},\ \frac{b'}{b}-\frac{c'}{c},\ \frac{c'}{c}-\frac{a'}{a}.$$

38. Shew that one of the cross-ratios of the pencil joining any point on the conic

$$S+\lambda S' \equiv (ax^2+by^2+cz^2)+\lambda(a'x^2+b'y^2+c'z^2)=0$$

to the four points given by $S=0$, $S'=0$ is

$$\frac{\dfrac{b+\lambda b'}{c+\lambda c'}}{\dfrac{ab'-a'b}{ac'-a'c}}.$$

Explain the result in case λ has one of the values

$$-\frac{a}{a'},\ -\frac{b}{b'},\ -\frac{c}{c'}.$$ (St John's.)

39. A rectangular hyperbola meets the axis of x in points K, L whose abscissae are λ, λ' respectively, and meets the axis of y in points M, N whose ordinates are μ, μ' respectively. Prove that the cross-ratio $P[KLMN]$, P being any other point on the hyperbola, is

$$\frac{(\lambda-\lambda')(\mu-\mu')\cos\omega}{\lambda\lambda'+\mu\mu'-(\lambda\mu'+\lambda'\mu)\cos\omega},$$

the axis being two lines inclined at an angle ω.

(Clare etc.)

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